α-COMPACTNESS IN BIFUZZY TOPOLOGICAL SPACES

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Abstract: Compact fuzzy topological spaces were first introduced by Chang [3]. α-compact fuzzy topological spaces were introduced by Gantner et al. [4]. Abd El-Monsef et al. [1] introduced the notion of α-almost compact fuzzy topological spaces. Mashhour et al. [8] introduced the notion of bifuzzy topological spaces and defined the notion of α-compactness in those spaces.

In this paper, we obtain some new results about α-compact bifuzzy topological spaces. We introduce and study the notion of pairwise α-almost compact bifuzzy topological spaces.

Keywords: Pairwise α-compact bfts; pairwise α-almost compact bfts; semi α-compact bfts; semi α-almost compact bfts; λ-adjoint fuzzy topology; pairwise Hausdorff bfts; twice OF-continuous mappings; twice almost fuzzy continuous mappings.

1. Preliminaries

Throughout this paper I will denote the unit interval [0, 1] of the real line. X and Y will be sets, s will be an arbitrary member of the index set S. I^X denotes the collection of all mappings from X into I. A member λ of I^X is called a fuzzy set of X [12]. The union ∪\λ_s and the intersection ∩\λ_s of the family {\λ_s} of fuzzy sets of X is defined to be the mapping sup \λ_s and inf \λ_s, respectively. For any two members λ and μ of I^X, λ ⩾ μ if and only if λ(x) ⩾ μ(x) for each x ∈ X, and in this case λ is said to contain μ, or μ is said to be contained in λ. 0 and 1 denote constant mappings taking all of X to 0 or 1, respectively. The complement λ' of fuzzy set λ of X is 1 − λ, defined by (1 − λ)(x) = 1 − λ(x), for each x ∈ X. Let f : X → Y be a mapping. If λ is a fuzzy set of X, f(λ) is defined as

\[ f(λ)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} λ(x) & \text{if } f^{-1}(y) \neq ∅, \\ 0 & \text{otherwise,} \end{cases} \]

for each y ∈ Y, and if μ is a fuzzy set of Y, f^{-1}(μ) is defined as

\[ f^{-1}(μ)(x) = μ(f(x)), \quad \text{for each } x ∈ X \text{ (cf. [3])}. \]

A fuzzy topology on a set X is a collection τ of I^X which contains the fuzzy sets 1, 0 and is closed under arbitrary unions and finite intersections [3]. The pair (X, τ) is called a fuzzy topological spaces (or fts, for short). Members of τ are fuzzy open sets. A fuzzy set λ of X is fuzzy closed if its complement λ' is fuzzy
open. The closure and the interior operators ( )\(^{-}\), ( )\(^{o}\) are defined in fts's in analogy with topological spaces [6].

The crisp subsets of \(X\) are just the \(\{0, 1\}\)-valued functions, i.e. the characteristic functions of the subsets of \(X\). If \(A\) is a subset of \(X\) and if \(\tau\) is a fuzzy topology on \(X\), then the set of restrictions \(\tau_A = \{\lambda \mid A : \lambda \in \tau\}\) is a fuzzy topology on \(A\), and \((A, \tau_A)\) is called a subspace of \((X, \tau)\) [4].

Let \((X, \tau)\) be a fts and \(\alpha \in \ell\); a collection \(U \subseteq I^X\) is said to be an \(\alpha\)-shading of \(X\) [4] if for each \(x \in X\), there is \(\lambda \in U\) with \(\lambda(x) > \alpha\). A subcollection \(\rho\) of an \(\alpha\)-shading \(U\) of \(X\) that is also an \(\alpha\)-shading is called an \(\alpha\)-subshading of \(U\).

Let \(X\) be a non-empty set, \(\tau_1\) and \(\tau_2\) be two fuzzy topologies on \(X\). The triple \((X, \tau_1, \tau_2)\) is called a bifuzzy topological space (bfts for short [8]).

### 2. \(\alpha\)-compactness in bifuzzy topological spaces

**Definition 2.1** [8]. An \(\alpha\)-shading \(U\) of \(X\) is said to be a pairwise fuzzy open \(\alpha\)-shading if \(U \subseteq \tau_1 \cup \tau_2\) and it contains at least one non-empty member of \(\tau_1\) and at least one non-empty member of \(\tau_2\).

**Definition 2.2** [8]. A bfts \((X, \tau_1, \tau_2)\) is pairwise \(\alpha\)-compact if every pairwise fuzzy open \(\alpha\)-shading of \(X\) has a finite pairwise fuzzy open \(\alpha\)-subshading.

We introduce:

**Definition 2.3.** An \(\alpha\)-shading \(U\) of \(X\) is said to be a \(\tau_1\tau_2\)-fuzzy open \(\alpha\)-shading if \(U \subseteq \tau_1 \cup \tau_2\).

**Definition 2.4.** A bfts \((X, \tau_1, \tau_2)\) is semi \(\alpha\)-compact if every \(\tau_1\tau_2\)-fuzzy open \(\alpha\)-shading of \(X\) has a finite \(\alpha\)-subshading.

**Definition 2.5.** Let \((X, \tau_1, \tau_2)\) be a fts; \(A \subseteq X\) is said to be a pairwise \(\alpha\)-compact (semi \(\alpha\)-compact) subset if every pairwise fuzzy open \(\alpha\)-shading of \(A\) has a finite \(\alpha\)-subshading of \(A\). \(A \subseteq X\) is said to be a pairwise \(\alpha\)-compact subset if \((A, \tau_1A, \tau_2A)\) is pairwise \(\alpha\)-compact (semi \(\alpha\)-compact).

**Proposition 2.1.** Let \((X, \tau_1, \tau_2)\) be a bfts and \(A\) be a crisp subset of \(X\); \(A\) is a semi \(\alpha\)-compact (pairwise \(\alpha\)-compact) subset iff \(A\) is a semi \(\alpha\)-compact (pairwise \(\alpha\)-compact) subspace.

**Proof.** Let \(A \subseteq X\) be a semi \(\alpha\)-compact subset and \(U = \{\lambda_s : s \in S, \lambda_s \in \tau_1 \cup \tau_2\}\) be a \(\tau_1A\tau_2A\)-fuzzy open \(\alpha\)-shading of \(A\). Then \(\hat{U} = \{\lambda_s : \lambda_s : A \in U\}\) is a \(\tau_1\tau_2\)-fuzzy open \(\alpha\)-shading of \(A\) and so there exists a finite subset \(S_0\) of \(S\) such that \(\{\lambda_s : \lambda_s : A \in U, s \in S_0\}\) is an \(\alpha\)-shading of \(A\). Then \(\{\lambda_s : A : s \in S_0\}\) is a finite \(\alpha\)-subshading of \(U\). Conversely, let \(A \subseteq X\) be a semi \(\alpha\)-compact subspace and \(V = \{\lambda_s : \lambda_s \in \tau_1 \cup \tau_2, s \in S\}\) be a \(\tau_1\tau_2\)-fuzzy open \(\alpha\)-shading of \(A\). Then \(V = \{\lambda_s : A : \lambda_s \in V\}\) is a \(\tau_1A\tau_2A\)-fuzzy open \(\alpha\)-shading of \(A\). Since \(A\) is a semi
α-compact subspace, there is a finite subset $S_0$ of $S$ such that $\{\lambda_s | A : \lambda_s \in V, s \in S_0\}$ is an α-shading of $A$. Then $\{\lambda_s : s \in S_0\}$ is a finite α-subshading of $V$ and so $A$ is a semi α-compact subset.

**Theorem 2.1.** The bfts $(X, \tau_1, \tau_2)$ is semi α-compact iff it is pairwise α-compact, $\tau_1$ α-compact and $\tau_2$ α-compact.

**Proof.** Necessity follows immediately from the observation that any pairwise fuzzy open, or $\tau_1$-fuzzy open or $\tau_2$-fuzzy open α-shading of $X$ is a $\tau_1\tau_2$-fuzzy open α-shading. Conversely, if a $\tau_1\tau_2$-fuzzy open α-shading is not pairwise fuzzy open, then it is $\tau_1$-fuzzy open or $\tau_2$-fuzzy open.

The following example shows that a bfts $(X, \tau_1, \tau_2)$ may be pairwise α-compact but not semi α-compact.

**Example 2.1.** Let $X$ be the set of all real numbers,

$$\tau_1 = \{\lambda \in \mathcal{P}(X) : \lambda = 1 \text{ or } \lambda \leq 0.8\}, \quad \tau_2 = \{1, 0, \nu\},$$

$$\nu(X) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise}. \end{cases}$$

For $\alpha = 0.7$, $(X, \tau_1, \tau_2)$ is pairwise α-compact because any pairwise fuzzy open α-shading of $X$ must contain $\nu$, or 1 but $X$ is not semi α-compact since $U = \{\text{all fuzzy points with values 0.8}\}$ is a $\tau_1$-fuzzy open and hence $\tau_1\tau_2$-fuzzy open α-shading which has no finite subshading.

**Theorem 2.2.** Let $F$ be a $\tau_1$-closed crisp subset of the bfts $(X, \tau_1, \tau_2)$. If $X$ is pairwise α-compact, then $F$ is a $\tau_2$ α-compact subspace.

**Proof.** Let $U$ be a $\tau_{2F}$-fuzzy open α-shading of $F$, where $\tau_{2F}$ is the $\tau_2$-relative topology on $F$, and let $V = \{\lambda \in \tau_2 : \lambda \mid F \in U\}$. Then $V \cup \{F'\}$ is a pairwise fuzzy open α-shading of $X$. So $V \cup \{F'\}$ has a finite α-subshading $\{\lambda_1, \lambda_2, \ldots, \lambda_n, F'\}$ of $X$. Then $\{\lambda_1 \mid F, \ldots, \lambda_n \mid F\}$ is a finite α-subshading of $F$. Hence $F$ is $\tau_{2F}$ α-compact.

**Definition 2.6.** If $\tau$ is a fuzzy topology on $X$ and $\lambda$ is a non-empty fuzzy subset in $X$, then the $\lambda$-adjoint fuzzy topology is given by

$$\tau(\lambda) = \{0, 1\} \cup \{\lambda \cup \mu : \mu \in \tau\}.$$

**Theorem 2.3.** A bfts $(X, \tau_1, \tau_2)$ is pairwise α-compact iff for each non-empty fuzzy set $\lambda$ in $\tau_1$, the adjoint fuzzy topology $\tau_2(\lambda)$ is α-compact and for each $\mu \in \tau_2$, $\tau_1(\mu)$ is α-compact.

**Proof.** Let $(X, \tau_1, \tau_2)$ be pairwise α-compact and $\lambda$ be a non-empty $\tau_1$-fuzzy open set and $U = \{\lambda \cup \mu_s : s \in S, \mu_s \in \tau_2\}$ be a $\tau_2(\lambda)$-fuzzy open α-shading of $X$. Then $\{\lambda\} \cup \{\mu_s\}$ is a pairwise open α-shading. So there exists a finite subset $S_0$ of
\[ \{ \lambda \} \cup \{ \mu_s : s \in S_0 \} \] is an \( \alpha \)-shading, so that \( \{ \lambda \cup \mu_s : s \in S_0 \} \) is a finite \( \alpha \)-subshading of \( U \), and hence \( (X, \tau_2(\lambda)) \) is \( \alpha \)-compact. Conversely, let \( U \) be a pairwise fuzzy open \( \alpha \)-shading of \( X \), \( U_1 = \{ \lambda_s : s \in S \} \) be the \( \tau_1 \)-fuzzy open sets of \( U \) and \( U_2 = \{ \mu_m : m \in M \} \) be the \( \tau_2 \)-fuzzy open sets of \( U \). \( \lambda = \bigcup \lambda_s \lambda_s \) is a \( \tau_1 \)-fuzzy open non-empty subset in \( X \). The family \( \{ \lambda_s, \mu_m : s \in S, m \in M \} \) is a pairwise fuzzy open \( \alpha \)-shading of \( X \). Then \( V = \{ \lambda \cup \mu_m : m \in M \} \) is a \( \tau_2 \)-fuzzy open \( \alpha \)-shading.

We introduce the following definition.

**Definition 2.7.** A bfts \( (X, T_1, T_2) \) will be called pairwise Hausdorff if \( x, y \in X \) and \( x \neq y \) imply that there exist \( \lambda \in T_1, \mu \in T_2 \) with \( \lambda(x) = \mu(y) = 1 \) and \( \lambda \cap \mu = 0 \).

**Theorem 2.4.** Let \( F \) be a crisp subset of a pairwise Hausdorff bfts \( (X, T_1, T_2) \). If \( F \) is a \( T_1 \)-\( \alpha \)-compact subspace for some \( \alpha \in [0, 1) \), then \( F \) is \( T_2 \)-fuzzy closed in \( X \).

**Proof.** Let \( x \in F' \), we will show that there exists \( \lambda \in T_2 \) with \( \lambda(x) = 1 \) and \( \lambda \approx F' \). For each \( y \in F \) we can find \( \lambda_y \in T_2 \) and \( \mu_y \in T_1 \) with \( \lambda_y(x) = \mu_y(y) = 1 \) and \( \lambda_y \cap \mu_y = 0 \). Thus \( \{ \mu_y : y \in F \} \) is a \( \tau_{1\mu} \)-fuzzy open \( \alpha \)-shading of \( F \), so it has a finite \( \alpha \)-subshading \( \{ \mu_{y_1}, \ldots, \mu_{y_n} : y \in F \} \). Let \( \lambda = \lambda_{y_1} \cap \cdots \cap \lambda_{y_n} \); thus \( \lambda(x) = 1 \) and \( \lambda \cap (\mu_{y_1} \cup \cdots \cup \mu_{y_n}) = 0 \). For each \( z \in F \), there exists a \( k \in \{1, 2, \ldots, n\} \) with \( \mu_{y_k}(z) > \alpha \approx 0 \). So \( \lambda(z) = 0 \), and hence \( \lambda \approx F' \), so that \( F' \) is \( \tau_2 \)-fuzzy open. Hence \( F \) is \( \tau_2 \)-fuzzy closed.

**Definition 2.8** \([3]\). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a mapping from a fuzzy space \( X \) to another space \( Y \). \( f \) is called a fuzzy continuous mapping if \( f^{-1}(\lambda) \in \tau \) for each \( \lambda \in \sigma \).

**Theorem 2.5.** Let \( (X, \tau_1, \tau_2) \) and \( (Y, \sigma_1, \sigma_2) \) be two bfts’s and \( f : (X, \tau_2) \rightarrow (Y, \sigma_2) \) be fuzzy continuous. If \( X \) is pairwise \( \alpha \)-compact, \( Y \) is pairwise Hausdorff and \( F \) is a \( \tau_1 \)-closed crisp subset in \( X \), then \( f(F) \) is a \( \sigma_1 \)-closed crisp subset in \( Y \).

**Proof.** Since \( F \) is a \( \tau_1 \)-closed crisp subset in \( X \), it follows that \( (F, \tau_{2\epsilon}) \) is \( \alpha \)-compact by Theorem 2.2, and then \( (f(F), \sigma_{2\epsilon}(F)) \) is \( \alpha \)-compact \([4]\). Since \( Y \) is pairwise Hausdorff, \( f(F) \) is a \( \sigma_1 \)-closed crisp subset by Theorem 2.4.
Example 2.2. Let $X = \{a, b, c, d\}$ and $\alpha A (\alpha \in I, A \subset X)$ denotes the fuzzy set in $X$ sending all of $A$ to $\alpha$ and $X - A$ to 0 and

$$
\lambda_1 = \frac{1}{2}1_{(a,b,c)}, \quad \lambda_2 = \left(\frac{1}{2}1_c\right) \vee 1_b, \quad \lambda_3 = \left(\frac{1}{2}1_{(a,b,c)}\right) \vee 1_b,
$$
$$
\lambda_4 = \frac{1}{2}1_b, \quad \lambda_5 = \left(\frac{1}{2}1_b\right) \vee 1_c,
$$

$\tau_1 = \{1, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\tau_2 = \{1, 0, \lambda_5\}$.

It is clear that $(X, \tau_1, \tau_2)$ is pairwise $\alpha$-compact for any $\alpha \in I$. Let $Y = \{x, y, z\}$ and

$$
\mu_1 = 1_x, \quad \mu_2 = 1_y, \quad \mu_3 = 1_z, \quad \mu_4 = 1_{(x,z)},
$$
$$
\mu_5 = 1_{(x,z)}, \quad \mu_6 = 1_{(y,z)}, \quad \mu_7 = \left(\frac{1}{2}1_z\right) \vee 1_y,
$$
$$
\mu_8 = \left(\frac{1}{2}1_z\right) \vee 1_x, \quad \mu_9 = \left(\frac{1}{2}1_z\right) \vee 1_{(x,y)}, \quad \mu_{10} = \frac{1}{2}1_z
$$

and

$$
\sigma_1 = \{1, 0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\},
$$
$$
\sigma_2 = \{1, 0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9, \mu_{10}\}.
$$

It is easy to see that $(Y, \sigma_1, \sigma_2)$ is pairwise Hausdorff. Let $f : X \to Y$ such that $f(a) = f(b) = f(c) = f(d) = x$. Clearly $f : (X, \tau_2) \to (Y, \sigma_2)$ is fuzzy continuous. Let $\lambda = \frac{1}{2}1_{(a,c,d)}; \lambda = \lambda_3$ be $\tau_1$-fuzzy closed and not crisp. Then $f(\lambda) = \frac{1}{2}1_x$ is not $\sigma_1$-closed.

3. Pairwise $\alpha$-almost compactness

**Definition 3.1.** A bfts $(X, \tau_1, \tau_2)$ is said to be pairwise $\alpha$-almost compact if every pairwise fuzzy open $\alpha$-shading $U$ of $X$ has a finite pairwise fuzzy open subcollection, the closure of whose members is an $\alpha$-shading.

**Definition 3.2.** A bfts $(X, \tau_1, \tau_2)$ is said to be semi $\alpha$-almost compact if every $\tau_1, \tau_2$-fuzzy open $\alpha$-shading $U$ of $X$ has a finite subcollection, the closure of whose members is an $\alpha$-shading.

**Definition 3.3** [1]. A fts $(X, \tau)$ is said to be regular if for each point $x \in X$ and each open neighbourhood $\lambda$ of $x$ there exists $\mu \in \tau$ with $\mu(x) = \lambda(x) \neq 0$ and $\mu \leq \lambda$.

**Definition 3.4** [5]. A fts $(X, \tau)$ is regular (R) if every fuzzy open set $U$ is a supremum of fuzzy open sets whose closure is less than $U$.

It is clear that every fuzzy regular space in the sense of Abd-Monsef–Ghanim is regular in the sense of Hutton–Reilly. The following example shows that the converse may be not true.

**Example 3.1.** Let $X = \{a, b\}, \lambda_1 = \frac{1}{2}1_{(a,b)}$, $\lambda_2 = \frac{1}{2}1_{(a,b)}$, $\lambda_3 = \frac{1}{2}1_{(a,b)}$, $\lambda_4 = \frac{1}{2}1_{(a,b)}$ and $\tau = \{1, 0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. It is easy to see that $(X, \tau)$ is regular in the sense of Hutton–Reilly but not regular in the sense of Abd El-Monsef–Ghanim.
Theorem 3.1 If \((X, \tau_i), i \in \{1, 2\}\), is a regular fts, then \((X, \tau_1, \tau_2)\) is pairwise (or semi) \(\alpha\)-compact iff it is pairwise (resp. semi) \(\alpha\)-almost compact.

Proof. Necessity follows immediately.

Conversely, let \(U\) be a pairwise fuzzy open \(\alpha\)-shading of \(X\). For any point \(x \in X\), there is \(\lambda_x \in U\) with \(\lambda_x(x) > \alpha\). Since \((X, \tau_i)\) is regular there is \(\mu_x \in \tau_1 \cup \tau_2\) such that \(\mu_x(x) = \lambda_x(x)\) and \(\mu_x \leq \lambda_x\). Since \(U\) is a pairwise fuzzy open \(\alpha\)-shading, \(\{\mu_x : x \in X\}\) is a pairwise fuzzy open \(\alpha\)-shading and so there exists a finite subset \(Y\) of \(X\) such that \(\{\mu_x : x \in Y\}\) is an \(\alpha\)-shading of \(X\). Therefore \(\{\lambda_x : x \in Y\}\) is a finite \(\alpha\)-subshading of \(U\) and so \(X\) is pairwise \(\alpha\)-compact.

It is easy to see that Theorem 3.1 does not hold for the fuzzy regular space in the sense of Hutton–Reilly.

Definition 3.5 [4]. Let \(X\) be a non-empty set and \(\alpha \in I\). A collection \(\mathcal{F}\) of fuzzy sets in \(X\) is said to be \(\alpha\)-centered if for all finite collections \(\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{F}\), there exists \(x \in X\) with \(\mu_k(x) \geq 1 - \alpha\) for all \(k \in \{1, 2, \ldots, n\}\).

Definition 3.6. Let \(\lambda\) be a fuzzy set in a fts \((X, \tau)\), then \(\lambda\) is called a fuzzy preopen [7] (resp. regular open [2], regular closed) set if \(\lambda \leq \lambda^\circ\) (resp. \(\lambda = \lambda^\circ\), \(\lambda = \lambda^\circ\)).

Theorem 3.2. For a bfts \((X, \tau_1, \tau_2)\), the following statements are equivalent:

(i) \(X\) is pairwise (or semi) \(\alpha\)-almost compact.

(ii) For every pairwise (or \(\tau_1\tau_2\)) fuzzy regular open \(\alpha\)-shading \(U = \{\lambda_s : s \in S\}\) of \(X\), there exists a finite subset \(S_0\) of \(S\) such that \(\{\lambda_s : s \in S_0\}\) is an \(\alpha\)-shading where \(\lambda_s\) means \(\tau_i - \text{cl}\) whenever \(\lambda_s \in \tau_i\), \(i \in \{1, 2\}\).

(iii) For every pairwise (or \(\tau_1\tau_2\)) fuzzy preopen \(\alpha\)-shading \(U = \{\lambda_s : s \in S\}\) of \(X\), there exists a finite subset \(S_0\) of \(S\) such that \(\{\lambda_s : s \in S_0\}\) is an \(\alpha\)-shading.

(iv) For every pairwise (or \(\tau_1\tau_2\)) fuzzy open \(\alpha\)-centered collection \(\mathcal{F}\), there is \(x \in X\) such that \(\lambda_{\chi(x)} \geq 1 - \alpha\) for all \(\lambda \in \mathcal{F}\).

Proof. (i) \(\Rightarrow\) (ii). Follows since every pairwise fuzzy regular open \(\alpha\)-shading is a pairwise fuzzy open \(\alpha\)-shading.

(ii) \(\Rightarrow\) (iii). Let \(U = \{\lambda_s : s \in S\}\) be a pairwise fuzzy preopen \(\alpha\)-shading of \(X\). For every \(\lambda_s \in U\), \(\lambda_s \leq \lambda_s^\circ\), the set \(\{\lambda_s^\circ : s \in S\}\) is a pairwise fuzzy regular open \(\alpha\)-shading. So there exists a finite subset \(S_0\) of \(S\) such that \(\{\lambda_s^\circ : s \in S_0\}\) is an \(\alpha\)-shading of \(X\), and since \(\lambda_s^\circ \leq \lambda_s^- = \lambda_s^\circ\), it follows that \(\{\lambda_s^- : s \in S_0\}\) is an \(\alpha\)-shading.

(iii) \(\Rightarrow\) (i). This is obvious.

(i) \(\Rightarrow\) (iv). Let \(\mathcal{F} \subset \mathcal{I}^X\) be a pairwise fuzzy open \(\alpha\)-centered collection. Assume that for each point \(x \in X\) there is \(\lambda_x \in \mathcal{F}\) such that \(\lambda_x^-(x) < 1 - \alpha\). Then \(U = \{\lambda_x^- : x \in X\}\) is a pairwise fuzzy open \(\alpha\)-shading of \(X\). Consequently, there is a finite subset \(Y \subset X\) such that \(\{\lambda_y^- : y \in Y\} = \{\lambda_y^\circ : y \in Y\}\) is an \(\alpha\)-shading of \(X\).
So, for each \( x \in X \), there is \( y \in Y \) such that \( \lambda_y^{\circ}(x) > \alpha \) and so \( \lambda_y^{-\circ}(x) < 1 - \alpha \), which implies that \( \lambda_x(x) = \lambda_y^{\circ}(y) < 1 - \alpha \); this contradicts that \( \mathcal{F} \) is \( \alpha \)-centered.

(iv) \( \Rightarrow \) (i). Let \( X \) satisfy (iv) but be not pairwise \( \alpha \)-almost compact. There is a pairwise fuzzy open \( \alpha \)-shading \( U \) of \( X \) such that for each finite subcollection \( U_0 \) of \( U \), \( \{ \lambda^{-\circ} : \lambda \in U_0 \} \) is not an \( \alpha \)-shading of \( X \). Hence for every finite subcollection \( U_0 \) of \( U \) there is a point \( x \in X \) such that \( \lambda^{-\circ}(x) < \alpha \) for all \( \lambda \in U_0 \), and then \( \{ \lambda^{-\circ} : \lambda \in U \} \) is a pairwise fuzzy open \( \alpha \)-centered. By (vi), there exists a point \( x_0 \in X \) such that \( \lambda^{-\circ}(x_0) > 1 - \alpha \) for all \( \lambda \in U \). Thus for all \( \lambda \in U \), we have \( \lambda'(x_0) > 1 - \alpha \) and this contradicts that \( U \) is an \( \alpha \)-shading. This completes the proof.

**Definition 3.7** [1]. A mapping \( f: X \to Y \), where \( X \) and \( Y \) are fts’s, is said to be **\( \theta F \)-continuous** if for each point \( x \in X \) and for every neighbourhood \( \mu \) of \( f(x) \), there exists a neighbourhood \( \lambda \) of \( x \) with \( \lambda(x) = [f^{-1}(\mu)](x) \) and \( f(\lambda^{-\circ}) \leq \mu^{-\circ} \).

**Definition 3.8** [2]. A mapping \( f: X \to Y \) where \( X \) and \( Y \) are fts’s is said to be **almost \( F \)-continuous** if the inverse image of every fuzzy regular open set in \( Y \) is fuzzy open in \( X \).

**Lemma 3.3.** A mapping \( f: X \to Y \) where \( X, Y \) are fts’s is almost \( F \)-continuous iff the inverse image of every fuzzy regular closed set in \( Y \) is fuzzy closed in \( X \).

**Proof.** It is obvious.

**Definition 3.9** [1]. A mapping \( f: X \to Y \), where \( X \) and \( Y \) are fts’s, is said to be **strong \( F \)-continuous** if for each \( x \in X \) and for every neighbourhood \( \mu \) of \( f(x) \), there exists a neighbourhood \( \lambda \) of \( x \) with \( \lambda(x) = [f^{-1}(\mu)](x) \) and \( f(\lambda^{-\circ}) \leq \mu^{-\circ} \).

We introduce:

**Definition 3.10.** Let \( X, Y \) be bfts’s; a mapping \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be **twice \( \theta F \)-continuous** if \( f: (X, \tau_i) \to (Y, \sigma_i) \) is \( \theta F \)-continuous, \( i = 1, 2 \).

**Definition 3.11.** If \( X, Y \) are bfts’s, a mapping \( f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is said to be **twice strong \( F \)-continuous** if \( f: (X, \tau_i) \to (Y, \sigma_i) \) is strong \( F \)-continuous, \( i \in \{1, 2\} \).

**Definition 3.12.** If \( (X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2) \) are bfts’s, a mapping \( f: X \to Y \) is said to be **twice almost fuzzy continuous** if \( f: (X, \tau_i) \to (Y, \sigma_i) \) is almost fuzzy continuous, \( i = 1, 2 \).

One may notice the following implications:

- twice \( \theta F \)-continuous
- twice strong \( F \)-continuous \( \Rightarrow \) twice fuzzy continuous
- twice almost \( F \)-continuous
Theorem 3.4. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a surjective twice \( \Theta F \)-continuous mapping, where \( X, Y \) are bfts's. If \( X \) is pairwise (or semi) \( \alpha \)-almost compact, so is \( Y \).

Proof. Let \( U \) be a pairwise fuzzy open \( \alpha \)-shading of \( Y \). For each \( y \in Y \), there is \( \mu_y \in U \) such that \( \mu_y(y) > \alpha \). Since \( f \) is twice \( \Theta F \)-continuous, for each \( y \in Y \) and for each \( x \in f^{-1}(y) \), there is \( \lambda_x \in \tau_1 \cup \tau_2 \) such that \( \lambda_x(x) = (f^{-1}(\mu_y))(x) \) and \( f(\lambda_x^-) \leq \mu_y^- \). The collection \( \{ \lambda_x : x \in X \} \) is a pairwise fuzzy open \( \alpha \)-shading of \( X \), and so, there is a finite subset \( Z \subset X \) such that \( \{ \lambda_x^- : z \in Z \} \) is an \( \alpha \)-shading of \( X \). Then \( \{ f(\lambda_x^-) : z \in Z \} \) is an \( \alpha \)-shading of \( Y \). Consequently, \( \{ \mu_y^- : y = f(x) \text{ for some } z \in Z \} \) is a finite \( \alpha \)-shading of \( Y \).

Theorem 3.5. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be a surjective twice strong \( F \)-continuous mapping and \( X, Y \) be bfts's. If \( X \) is pairwise (or semi) \( \alpha \)-almost compact, then \( Y \) is pairwise (resp. semi) \( \alpha \)-compact.

Proof. Follows by the same arguments as in the above theorem.

Theorem 3.6. Let \( f : X \to Y \) be a surjective twice almost fuzzy continuous mapping. If \( X \) is pairwise (or semi) \( \alpha \)-almost compact, so is \( Y \).

Proof. Let \( U \) be a pairwise fuzzy regular open \( \alpha \)-shading of \( Y \). For each \( y \in Y \), there is \( \mu_y \in U \) such that \( \mu_y(y) > \alpha \). Since \( f \) is twice almost fuzzy continuous, \( \{ f^{-1}(\mu_y) : y \in Y \} \) is a pairwise fuzzy open \( \alpha \)-shading of \( X \). As \( X \) is pairwise \( \alpha \)-almost compact, there is a finite subset \( Z \subset Y \) such that \( \{ f^{-1}(\mu_z) : z \in Z \} \) is an \( \alpha \)-shading of \( X \). Since \( f \) is twice almost fuzzy continuous and each \( \mu_y^- \) is fuzzy regular closed, \( (f^{-1}(\mu_z))^- \leq f^{-1}(\mu_z^-) \) for each \( z \in Z \). Hence \( \{ f^{-1}(\mu_z^-) : z \in Z \} \) is an \( \alpha \)-shading of \( X \). Consequently, \( \{ \mu_y^- : z \in Z \} \) is a finite \( \alpha \)-shading of \( Y \).

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References


