IN THIS PAPER we introduce some new separation axioms based on Aull and Thron axioms, $R_o$ and semi-$R_o$ axioms. Also we introduce and study separation axioms based on Mashhour axioms.

1. Introduction

The first separation axiom between $T_0$ and $T_1$ was introduced by Yongs (1). Later several axioms between $T_0$ and $T_1$ were introduced by Aull and Thron (2) using the notion of the derived set of points. Shanin (3) and Davis (4) discovered the $R_o$ axiom which is weaker than $T_1$ but independent of $T_0$. Semi-$R_o$ axioms were introduced and studied by Maheshwari and Prasad (5).

In 1983 Mashhour, and others (6) introduced and studied the notions of supraseparation axioms $s-T_i$ ($i = 0, 1, 2$).

Throughout this paper $(X, \tau^*)$ will denote a supratopological space (6) associated with a topology $\tau(\tau \subset \tau^*)$ on $X$. The elements of $\tau^*$ are called supr-open sets and their complements are called supraclosed sets. By sup. d $\{x\}$ and sup. cl $\{x\}$ we mean a supr-derived set and supr-closure set of $\{x\}$, which are defined in supratopological spaces in analogy with topological spaces.

In this paper we introduce new separation axioms based on Aull and Thron axioms (2) other axioms weaker than Mashhour axioms (7) and finally we introduce new separation axioms based on $T''_2$ (8).

2. New separation Axioms related to $s-T_0$ and $s-T_1$

In what follows we introduce new separation axioms based on Aull and Thron axioms (2) taking in consideration the behaviour of a supr-derived set (8).
Definition 2.1. A supraspace \((X, \tau^*)\) is called

(i) \(s-T_D\), if for every \(x \in X\), \(\sup d \{x\}\) is supraclosed.

(ii) \(s-T_Y\), if for every \(x, y \in X\), \(x \neq y\), \(\sup \cl \{x\} \cap \sup \cl \{y\}\) is degenerate.

(iii) \(s-T_{UD}\), if for \(x \in X\), \(\sup d \{x\}\) is a union of disjoint supraclosed sets.

(iv) \(s-T_{DD}\), \(T_D + \) for every \(x, y \in X\), \(x \neq y\), \(\sup d \{x\} \cap \sup d \{y\} = \emptyset\).

(v) \(s-T_{YS}\), if for every \(x, y \in X\), \(x \neq y\), \(\sup \cl \{x\} \cap \sup \cl \{y\}\) is either \(\emptyset\) or \(\{x\}\) or \(\{y\}\).

Theorem 2.1. For a supraspace \((X, \tau^*)\), the following statements hold.

(i) \(s-T_1 \rightarrow s-T_{DD}\)

(ii) \(s-T_{DD} \rightarrow s-T_{YS}\)

(iii) \(s-T_{UD} \rightarrow s-T_O\)

Proof. (i) Let \(X\) be an \(s-T_1\) space, then for every two distinct points \(x\) and \(y\) of \(X\), \(\{x\}\) and \(\{y\}\) are supraclosed. Therefore, \(\sup d \{x\} = \sup d \{y\} = \emptyset\) which implies \(X\) is \(s-T_D\). Also, \(\sup d \{x\} \cap \sup d \{y\} = \emptyset\). Hence \(X\) is \(s-T_{DP}\).

(ii) Let \(X\) be an \(s-T_{DP}\), then for every two distinct points \(x\) and \(y\), \(\sup d \{x\} \cap \sup d \{y\} = \emptyset\) implies \(\sup \cl \{x\} \cap \sup \cl \{y\} = \emptyset\). Therefore, \(X\) is \(s-T_{YS}\).

(iii) Let \(X\) be an \(s-T_{UD}\), and \(x \in X\), then \(\sup d \{x\}\) is a union of disjoint supraclosed sets. Therefore, \(X\) is \(s-T_O\).

Theorem 2.2.

A supraspace \((X, \tau^*)\) is \(s-T_D\) iff for each \(x \in X\), there exists a supraopen set \(U\) and a supraclosed set \(F\) such that \(\{x\} = U \cap F\).

Proof: Since $X$ is $s$-$T_1$ space, then $\text{sup. } d \{ x \}$ is a supraclosed set. Take $U = X - \text{sup. } d \{ x \}$ and $F = \text{sup. } cl \{ x \}$. Then $U \cap F = \{ x \}$. Conversely, let $x \in X$ be any point. We have $\text{sup. } d \{ x \} = \text{sup. } cl \{ x \} - \{ x \} = \text{sup. } cl \{ x \} - (U \cap F)$. But $F$ is a supraclosed set and $x \in F$ implies $\text{sup. } cl \{ x \} \subseteq F$. Then $\text{sup. } d \{ x \} = \text{sup. } cl \{ x \} - (U \cap \text{sup. } cl \{ x \}) = \text{sup. } cl \{ x \} \cap (X - U)$. Hence the set $\text{sup. } d \{ x \}$ can be written as the intersection of two supraclosed sets. Therefore, supraderived $\{ x \}$ is a supraclosed set and $X$ is $s$-$T_1$ space.

Now we introduce another axiom called $s$-$R_o$. This axiom is weaker than $s$-$T_1$ but independent of $s$-$T_0$.

Definition 2.2. A supraspace $(X, \tau^*)$ is said to be $s$-$R_o$ if for every supraclosed set $F$ and point $x \in F$, $F \cap \text{sup. } cl \{ x \} = \emptyset$.

Theorem 2.3.

The following statements are equivalent for a supraspace $(X, \tau^*)$.

(i) $X$ is $s$-$R_o$.

(ii) For each supraopen set $G$ and each point $x \in G$, $\text{sup. } cl \{ x \} \subseteq G$.

(iii) For any two distinct points $x, y \in X$, either $\text{sup. } cl \{ x \} = \text{sup. } cl \{ y \}$ or $\text{sup. } cl \{ x \} \cap \text{sup. } cl \{ y \} = \emptyset$.

Proof: (i) $\Rightarrow$ (ii). Let $X$ be an $s$-$R_o$-space and $G \subseteq X$ be a supraopen set containing $x \in X$. Then $X - G$ is a supraclosed get not containing $x$. Hence $\text{sup. } cl \{ x \} \cap (X - G) = \emptyset$ and $\text{sup. } cl \{ x \} \subseteq G$.

(ii) $\Rightarrow$ (iii). Let $x$ and $y$ be two distinct points of $X$, then either $x \in \text{sup. } cl \{ y \}$ or $y \in \text{sup. } cl \{ x \}$ implies $\text{sup. } cl \{ x \} = \text{sup. } cl \{ y \}$ or $x \in \text{sup. } cl \{ y \}$, implies $X - \text{sup. } cl \{ y \}$ is a supraopen set containing $x$. Hence $\text{sup. } cl \{ x \} \cap X - \text{sup. } cl \{ y \} = \emptyset$.

(iii) $\Rightarrow$ (i). Let $F \subseteq X$ be any supraclosed set and $x \in F$. Then for any $y \in F$, $\text{sup. } cl \{ x \} \cap \text{sup. } cl \{ y \} = \emptyset$ and so $F \cap \text{sup. } cl \{ x \} = \emptyset$. Therefore, $X$ is $s$-$R_o$ space.

Theorem 2.4.

A supraspace \((X, \tau^*)\) is \(s\)-\(R_o\) and \(s\)-\(T_o\) iff it is \(s\)-\(T_1\).

**Proof:** Let \(X\) be an \(s\)-\(R_o\) and \(s\)-\(T_o\), and let \(x, y\) be two distinct points of \(X\). Since \(X\) is \(s\)-\(T_o\), there exists a supraopen neighbourhood \(U_x\) of one point, say \(x\), to which \(y\) does not belong. So \(X - U_x\) is a supraclosed neighbourhood of \(y\) and \(x \in X - U_x\). Since \(X\) is \(s\)-\(R_o\), sup. cl \(\{x\} = (X - U_x) = \emptyset\) and \(y \in\) sup. cl \(\{x\}\). Hence there exists a supraopen neighbourhood \(U_y\) not containing \(x\).

Conversely, let \(X\) be an \(s\)-\(T_1\) space, then it is \(s\)-\(T_o\). Let \(F\) be any supraclosed subset of \(X\), \(x \in F\) be any point of \(X\). Since \(\{x\}\) is supraclosed, then sup. cl \(\{x\}\) \(F = \emptyset\) and \(X\) is \(s\)-\(R_o\) space.

3. New separation axioms based on Mashhour axioms

In what follows, new separation axioms are introduced by taking in consideration Mashhour axioms (7), the concept of supraopeness (6) and the boundary operator. These axioms are weaker than Mashhour axioms.

**Definition 3.1.** A supraspace \((X, \tau^*)\) is \(s\)-\(T'_o\) if for any two distinct points, there is a supraopen neighbourhood of one point to which the other is a boundary point.

**Definition 3.2.** A supraspace \((X, \tau^*)\) is \(s\)-\(T'_1\) if for any two distinct points \(x, y \in X\), there is a supraopen neighbourhood of one of them, say \(x\), to which \(y\) is a boundary point and an supraopen neighbourhood of \(y\) to which \(x\) does not belong.

**Definition 3.3.** A supraspace \((X, \tau^*)\) is \(s\)-\(T'_o\) if for any two distinct points \(x, y \in X\), there is a supraopen neighbourhood of \(x\) to which \(y\) is a boundary point.

From the above definitions one may deduce the implications between these new axioms and Mashhour axioms \(T'_o, T'_1\) and \(T''_1\) (7).

\[
\begin{array}{ccc}
T''_1 & \longrightarrow & T'_1 \\
\downarrow & & \downarrow \\
\longrightarrow & & \longrightarrow \\
\text{s-}T''_1 & \longrightarrow & \text{s-}T'_1 \\
\downarrow & & \downarrow \\
& \longrightarrow & \text{s-}T'_o
\end{array}
\]

The following examples show that the inverse of these implications are false, in general.

Example 3.1. Let \( X = \{ x, y, z, w \} \) with topology \( \tau = \left\{ X, \emptyset, \{ x \} \right\} \), then \( X \) is s-T" but it is not T".

Example 3.2. Let \( X = \{ x, y, z, w \} \) with topology \( \tau = \left\{ X, \emptyset, \{ x \}, \{ x, y \}, \{ x, z \}, \{ y, z \} \right\} \). Then \( X \) is s-T" but it is neither n-T' nor s-T".

Theorem 3.1.
An s-T" space which is s-T is s-T.

Proof: Let \( x \) and \( y \) be two distinct points in \( X \). Since \( X \) is s-T", there is a supraopen neighbourhood \( U \) of one of them, say \( x \), to which \( y \) is a boundary point. Since \( X \) is s-T, the singleton \( \{ x \} \) may be considered as a supraclosed subset of \( X \). Put \( V = X - \{ x \} \), \( V \) is a supraopen neighbourhood of \( y \) to which \( X \) does not belong. Therefore, \( X \) is s-T".

Theorem 3.2.
An s-T" space in which every non empty supraopen set is dense, is s-T".

Proof: Obvious.

Theorem 3.3.
A supraspace \( X \) is R_0 and s-T" iff it is s-T'.

Proof: Let \( x \) and \( y \) be two distinct points of \( X \) which is R_0 and s-T". Then there exists a supraopen neighbourhood \( U \) of one point, say \( x \), such that \( y \) is a boundary point, i.e. \( y \in \partial (U) \). Since \( b(U) \cap U = \emptyset \), \( x \in (b(U))^d \), so \( x \in \text{cl} (b(U)) \). Since \( X \) is an R_0-space, \( \text{cl} \{ x \} \cap \text{cl} (b(U)) = \emptyset \). Then there exists an open neighbourhood \( V = X - \text{cl} \{ x \} \) of \( x \) for which \( x \) does not belong. Hence \( X \) is s-T'. The converse follows directly from Theorem 2.4.

Remark. Since the intersection of open and supraopen sets need not be supraopen, then not every open (supraopen) subspace of s-T", s-T', and s-T" spaces is s-T", s-T', and s-T".
4. New supraseparation axioms based on $T''_2$

In this article, we introduce and discuss new supraseparation axioms: $s-T''_2$, $s-T''_2w$, $s-T'_2w$, $s-T_2$, and $s-T^*_2$. The definitions of these new axioms are based on the behaviour of the weak boundary of a set. A point $x$ of a space $X$ is a weak boundary point of $A \subset X$ if the closure of any neighbourhood $N_x$ of $x$ satisfies

$$\text{cl} \ (N_x) \cap A \neq \emptyset$$

and

$$\text{cl} \ (N_x) \cap (X-A) \neq \emptyset.$$

The set of all weak boundary points of a set $A$ is called a weak boundary set of $A$ and is denoted by $b^w(A)$.

**Definition 4.1.** A supraspace $(X, \tau^*)$ is $s-T''_2$ if, for any two distinct points $x$ and $y$ of $X$, there exist two supraopen neighbourhoods of $x$ and $y$ whose boundary sets are disjoint.

**Definition 4.2.** A supraspace $(X, \tau^*)$ is $s-T''_2w$ space if, for any two distinct points $x$ and $y$ of $X$, there exist two supraopen neighbourhoods of $x$ and $y$ whose weak boundary sets are disjoint.

**Definition 4.3.** A supraspace $(X, \tau^*)$ is $s-T'_2w$ space if for any two distinct points $x$ and $y$ of $X$, there exist two disjoint supraopen neighbourhoods of $x$ and $y$ whose boundary sets are disjoint.

**Definition 4.4.** A supraspace $(X, \tau^*)$ is $s-T_2$ space if, for any two distinct points $x$ and $y$ of $X$, there exist two supraopen neighbourhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $\text{cl} \ (U_x) \cap U_y = \emptyset$ and $U_x \cap \text{cl} \ (U_y) = \emptyset$, where $V = \{x : \text{cl} \ (U_x) \cap \text{cl} \ (V) \neq \emptyset \}$ for every supraopen neighbourhood $U_x$ of $x$.

**Definition 4.5.** A supraspace $(X, \tau^*)$ is $s-T^*_2$ if for any two distinct points $x$ and $y$ of $X$, there exists a supraopen neighbourhood $U_x$ of $x$ such that $U_x \cap Y \neq \emptyset$ does not contain $y$.

**Definition 4.6.** A supraspace $(X, \tau^*)$ is $s-T'_2$ space if, for any two distinct points $x$ and $y$ of $X$, there exist two supraopen neighbourhoods of $x$ and $y$ whose closures are disjoint.

Theorem 4.1.

The following statements are satisfied for a supraspace \((X, \tau^* )\).

(i) \(s-T''_2 \rightarrow s-T''_2 \).

(ii) \(s-T'_{2w} \rightarrow s-T'_{2} \).

Proof: Directly from the definitions.

Theorem 4.2.

For a supraspace \((X, \tau^* )\):

\[
\begin{align*}
s-T'_{2} & \quad s-T'_{2}^* & \quad s-T'_{2} & \quad s-T''_{2}.
\end{align*}
\]

Proof: Let \(X\) be an \(s-T'_{2}\) space and \(x,y\) two distinct points of \(X\). Then there exist two supraopen neighbourhoods \(U_x\) and \(U_y\) of \(x\) and \(y\), respectively, such that \(\text{cl}(U_x) \cap U_{**y} = \emptyset\) and \(U_x** \cap \text{cl}(U_y) = \emptyset\).

Then \(X\) is \(s-T'_{2}^*\). This implies \(x \in U_{**y}\), then there exists a supraopen neighbourhood \(U_x\) of \(x\) such that \(\text{cl}(U_x) \cap \text{cl}(U_y) = \emptyset\). So, \(X\) is \(s-T'_{2}\) and consequently \(s-T''_{2}\).

Theorem 4.3.

An \(s-T'_{2}\) which is also \(s-T'_{2w}\) is \(s-T'_{2w}\) supraspace.

Proof. Obvious.

Theorem 4.4.

A connected \(T_1\)-space is \(s-T''_{2}\).

Proof. Let \(X\) be a connected \(T_1\)-space, and let \(x,y\) be two distinct points of \(X\). Then \(X - \{y\}\) and \(X - \{x\}\) may be considered as two open and consequently two supraopen neighbourhoods of \(x\) and \(y\), respectively. Since \(X\) is connected, then \(\text{cl}(X - \{x\}) = \text{cl}(X - \{y\}) = X\). So, \(b(X - \{y\}) = \{y\}\) and \(b(X - \{x\}) = \{x\}\) implies \(X\) is \(s-T''_{2}\) space.

From the above discussion one may deduce the followin diagram.

References


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في هذا البحث قدمنا بعض穆سلمان الانفصال الجديدة التي تتمحى على
Muslims الانفصال التي قدمها أولا وإثبات عام 1972 وكذلك على مسلمات
R₀، R₀ النصفية. كذلك قدمنا مسلمات انفصال جديدة تستند على
تمك انفصال مشهور عام 1970 ودراسة خواص تلك المسلمات الجديدة
وارتباطها بعض المسلمين الأخرى.