GENERALIZED TOPOLOGICAL APPROACH FOR
ROUGH SETS

M. E. Abd El-Monsef, A. S. Salama and O. A. Embaby

ABSTRACT: The main objective of this paper is to propose a set-theoretic
framework for rough sets using mixed neighborhood systems. The concept of
Frechet (V) spaces and some types of neighborhood systems based on serial
binary relations are used. The approximations are constructed using (right,
left, and mixed) neighborhood systems. A comparison between these three
approaches is imposed.

1. INTRODUCTION

In 1994, Zadeh [16] has been pointed out the importance of information granulation.
According to Zadeh [17],

"Information granulation involves partitioning a class of objects (points) into granules,
with a granule being a clump (crisp/fuzzy subset) of objects which are drawn together
by indistinguishability, similarity or functionality."

In 1998, Lin [10,11] rephrased Zadeh’s words as follows:

"Information granulation is a collection of granules, with a granule being a clump of
objects (points) which are drawn towards an object. In other words, each object is
associated a (empty, finite or infinite) family of clumps."

Mathematically, the association of each object with such a family of clumps is
the notion of neighborhood system (NS). The notion of neighborhood system is
originated from a generalization of topological space which called Frechet (V) space
[9,15]. If all clumps are non-empty, then neighborhood system is a Frechet (V)
space [15]. If a neighborhood system space satisfies the supratopology axioms, it
defines a supra topological space, such neighborhood system space will be called
supratopological neighborhood system [13]. If a neighborhood system satisfies certain
axioms (topology axioms), it defines a topological space, such neighborhood system

will be called topological neighborhood system [5, 6, 15]. If the union of all clumps for all objects equals the set of objects, then NS is a covering [18]. If there is at most one clump per object, then neighborhood system is defined by binary relation and is called binary neighborhood system [10]. If the binary relation is an equivalence relation, then neighborhood system is a rough set system [14].

In neighborhood system, clumps are called neighborhoods. Using neighborhoods, one can define open sets, closed sets, and hence the interior and closure of any subset [6, 15]. If we take equivalence classes as neighborhoods, the lower and upper approximations in rough set theory are precisely the interior and closure respectively. From this point of view, rough set theory is a special form of neighborhood system theory [1, 5, 8, 9, 10, 11, 14]. Formly neighborhoods play the most fundamental role in mathematical analysis. It is a common notion in databases [3, 12], in rough sets [14], in logic [2], in texts of genetic algorithms [4], and many others.

2. PRELIMINARIES

We shall summarize some fundamental notions on a neighborhood system [9, 15].

Let \( U \) be a non-empty finite set and \( p \in U \), then:

1. A neighborhood of \( p \), denoted by \( N(p) \), is a non-empty subset of \( U \). \( N(p) \) may or may not contains \( p \).

2. A neighborhood system of \( p \), denoted by \( NS(p) \), is the maximal family of neighborhoods of \( p \). If \( p \) has no neighborhood, then \( NS(p) \) is an empty family.

3. A neighborhood system of \( U \), denoted by \( NS(U) \), is the collection of \( NS(p) \) for all \( p \) in \( U \).

4. A set \( U \) together with \( NS(U) \), \((U, NS(U))\), is called a neighborhood system space (NS-space) or simply neighborhood system.

**Definition 2:** Let \( R \subseteq U \times U \), \( R \) is called a serial binary relation if for each \( x \in U \) there is \( y \in U \) such that \( xRy \).

**Definition 3:** Let \((U, NS(U))\) be a neighborhood system space, then:

1. \( NS(U) \) is discrete, if \( NS(U) \) is the power set of \( U \).
2. $NS(U)$ is indiscrete, if $NS(U) = \{U\}$.
3. $NS(U)$ is serial, if $NS(p) \neq \emptyset$, $\forall p \in U$ (called Frechet $(V)$ space).
4. $NS(U)$ is reflexive, if $p \in N(p)$, $\forall p \in U$.
5. $NS(U)$ is symmetric, if $q \in N(p) \Rightarrow p \in N(q)$, $\forall p, q \in U$.
6. $NS(U)$ is transitive, if $r \in N(q)$ and $q \in N(p) \Rightarrow r \in N(p)$, $\forall p, q, r \in U$.
7. $NS(U)$ is Euclidean if $q \in N(p)$ and $r \in N(p) \Rightarrow r \in N(q)$, $\forall p, q, r \in U$.

Definition 4: [15] A set $U$ is a Frechet $(V)$ space (or briefly $(V)$ space) if with each element $p \in U$ there is associated a non-empty family of subsets of $U$. Each subset is called a neighborhood of $p$.

Example 1: The set of points in the plane is a $(V)$ space if a neighborhood of a point $p$ is taken to be, e.g., the interior of an arbitrary circle with center at $p$. Clearly, a neighborhood in this case can be defined in many ways, for instance, the interior and boundary of any square with center at $p$. Or we take, e.g., the neighborhood of $p$ is $\{p\}$.

Example 2: The set of all real functions of a real variable is a $(V)$ space, if a neighborhood of $f(x)$ is defined to be the set of all functions $g(x)$ which satisfy the inequality $|f(x) - g(x)| \leq c$ for all values of $x$, where $c$ is a given positive real number.

Proposition 1: [15] Let $U$ be a $(V)$ space and $A \subseteq U$, then:

(i) $A$ is open iff $A = \text{int}(A)$, where $\text{int}(A)$ is the interior of $A$.

(ii) $A$ is closed iff $A = \text{int}(A)$, where $\text{cl}(A)$ is the closure of $A$.

(iii) In a $(V)$ space, the union of a finite family of open sets is open and the intersection of a finite family of closed sets is closed.

Definition 5: [14] Let $U$ be a non-empty finite set and $R$ be an equivalence binary relation and $A \subseteq U$, then the lower approximation and the upper approximation of $A$ are defined respectively by

$$R(A) = \{p \in U \mid [p]_R \subseteq A\},$$

$$\bar{R}(A) = \{p \in U \mid [p]_R \cap A \neq \emptyset\},$$

where $[p]_R$ is the equivalence class of $p$. 

Generalized Topological Approach for Rough Sets
3. MIXED NEIGHBORHOOD APPROXIMATION SPACE AS A GENERALIZATION OF PAWLAK APPROXIMATION SPACE

Let $U$ be a non-empty finite set and $R$ be a serial binary relation.

**Definition 6:** The right neighborhood of an element $x \in U$ is the set
$$N_r(x) = \{y \in U | x R y\}.$$

**Definition 7:** The right neighborhood system of an element $x \in U$ is the class
$$NS_r(x) = \{N_r(x)\}.$$

**Example 3:** Let $U = \{a, b, c, d\}$, $R = \{(a, a), (b, a), (b, c), (c, c), (c, d), (d, b)\}$.
Then we have $N_r(a) = \{a\}, N_r(b) = \{a, c\}, N_r(c) = \{c, d\}, N_r(d) = \{b\}, NS_r(a) = \{\{a\}\}$,
$NS_r(b) = \{\{a, c\}\}, NS_r(c) = \{\{c, d\}\}, NS_r(d) = \{\{b\}\}.$

**Definition 8:** The left neighborhood of an element $x \in U$ is the set
$$N_l(x) = \{y \in U | y R x\}.$$

**Definition 9:** The left neighborhood system of an element $x \in U$ is the class
$$NS_l(x) = \{N_l(x)\}.$$

**Example 4:** According to Example 3, then we have $N_l(a) = \{a, b\}, N_l(b) = \{c\}, N_l(c) = \{d\}, N_l(d) = \{a\}$,
$NS_l(a) = \{\{a, b\}\}, NS_l(b) = \{\{c\}\}, NS_l(c) = \{\{d\}\}, NS_l(d) = \{\{a\}\}.$

**Definition 10:** The mixed neighborhood system of an element $x \in U$ is the class
$$NS_m(x) = \{N_r(x), N_l(x)\}.$$

**Definition 11:** The mixed neighborhood of an element $x \in U$ is denoted by
$N(x)$ such that $N(x) \in NS_m(x)$.

**Example 5:** According to Example 3, the mixed neighborhood systems are given by $N_m(a) = \{a\}, N_m(b) = \{a, b\}, N_m(c) = \{c\}, N_m(d) = \{d\}.$

**Definition 12:** Let $A \subseteq U$, the derived set of $A$ (denoted by $A'$) is defined by
$$A' = \{p \in U | \forall \forall N(p) \in NS(p), (N(p) - \{p\} \cap A = \emptyset\},$$
where $NS(p)$ is the neighborhood system of $p$ which may be defined as (right, left, or mixed) neighborhood system of $p$.

**Definition 13:** Let $A \subseteq U$, the sets $A'_r$, $A'_l$, and $A'_m$ are defined by
Generalized Topological Approach for Rough Sets

Definition 14: The class of closed sets is defined by
\[ \Gamma = \{ F \subseteq U \mid F' \subseteq F \}. \]

The class of open sets is defined by
\[ \Omega = \{ G \subseteq U \mid G = U - F, F \in \Gamma \}. \]

Definition 15: The classes of r-closed sets, l-closed sets, and m-closed sets are defined respectively by
\[ \Gamma_r = \{ F \subseteq U \mid F' \subseteq F \}, \]
\[ \Gamma_l = \{ F \subseteq U \mid F'_l \subseteq F \}, \]
and
\[ \Gamma_m = \{ F \subseteq U \mid F'_m \subseteq F \}. \]

Definition 16: The classes of r-open sets, l-open sets, and m-open sets are defined respectively by
\[ \Omega_r = \{ G \subseteq U \mid G = U - F, F \in \Gamma_r \}, \]
\[ \Omega_l = \{ G \subseteq U \mid G = U - F, F \in \Gamma_l \}, \]
and
\[ \Omega_m = \{ G \subseteq U \mid G = U - F, F \in \Gamma_m \}. \]

Remark 1: By Definitions 14, 15, and 16 it is obvious that any r-open set, l-open set, and m-open set is open and also any r-closed set, l-closed set, and m-closed set is closed.

Definition 17: The lower approximation of a set \( A \subseteq U \) is defined by
\[ \underline{R}(A) = \{ G \subseteq U \mid G \in \Omega, G \subseteq A \}. \]

The upper approximation of a set \( A \subseteq U \) is defined by
\[ \overline{R}(A) = \{ F \subseteq U \mid F \in \Gamma, A \subseteq F \}. \]

Definition 18: The lower approximations of a set \( A \subseteq U \) using (right, left, and mixed) neighborhood systems are defined respectively by
\[ \underline{R}_r(A) = \{ G \subseteq U \mid G \in \Omega_r, G \subseteq A \}, \]
\[ \underline{R}_l(A) = \{ G \subseteq U \mid G \in \Omega_l, G \subseteq A \}, \]
and
\[ \underline{R}_m(A) = \{ G \subseteq U \mid G \in \Omega_m, G \subseteq A \}. \]
Definition 19: The upper approximations of a set $A \subseteq U$ using (right, left, and mixed) neighborhood systems are defined respectively by
\[
\overline{R}_r(A) = \cap \{ F \mid F \in \Gamma_r, A \subseteq F \},
\]
\[
\overline{R}_l(A) = \cap \{ F \mid F \in \Gamma_l, A \subseteq F \},
\]
and
\[
\overline{R}_m(A) = \cap \{ F \mid F \in \Gamma_m, A \subseteq F \},
\]

Definition 20: Let $A \subseteq U$. The boundary set of $A$ is defined by
\[
B(A) = \overline{R}(A) - \overline{R}(A).
\]

Definition 15: Let $A \subseteq U$, the sets $B_r(A)$, $B_l(A)$, and $B_m(A)$ are defined respectively by
\[
B_r(A) = \overline{R}_r(A) - \overline{R}_r(A),
\]
\[
B_l(A) = \overline{R}_l(A) - \overline{R}_l(A),
\]
and
\[
B_m(A) = \overline{R}_m(A) - \overline{R}_m(A).
\]

Remark 2: $(U, R)$ will be called neighborhood approximation space (NAS) which is a generalization of Pawlak approximation space (PAS) [14] because in PAS, $R$ has been taken as an equivalence relation and we take $R$ as a serial binary relation. If we used mixed neighborhood systems, $(U, R)$ will be called mixed neighborhood approximation space (MNAS).

4. MAIN CONTRIBUTION

Lemma 1: Any $r$-closed set and $l$-closed set is $m$-closed.

Proof: Let $A \subseteq U$ be $r$-closed set, therefore $A' \subseteq U$ such that $A' = \{ p \in U \mid (N_r(p) - \{ p \}) \cap A \neq \emptyset \}$. Since $A'_r = \{ p \in U \mid (N_l(p) - \{ p \}) \cap A \neq \emptyset \}$, we have $A'_r \subseteq A'$. Hence $A'_r \subseteq A$ which implies $A$ is $m$-closed. Therefore any $r$-closed set is $m$-closed. Similarly we can prove that any $l$-closed set is $m$-closed.

Lemma 2: Any $r$-open set and $l$-open set is $m$-open.

Proof: Let $G \subseteq U$ be $r$-open set and $U - G = F$. So $F$ is $r$-closed set and from Lemma 1 we have $F$ is $m$-closed. Therefore $U - F = G$ is $m$-open. Hence any $r$-open set is $m$-open set. Similarly we can prove that any $l$-open set is $m$-open.
**Theorem 1:** Let \((U, R)\) be NAS and \(A\) be subset of \(U\), then:

(i) \(\mathcal{R}_r(A) \cup \mathcal{R}_l(A) \subseteq \mathcal{R}_m(A)\),

(ii) \(\overline{\mathcal{R}}_m(A) \subseteq \overline{\mathcal{R}}_r(A) \cap \overline{\mathcal{R}}_l(A)\),

(iii) \(B_m(A) \subseteq B_r(A) \cap B_l(A)\).

**Proof:** (i) From the definition of \(\mathcal{R}_r(A)\), we have \(\mathcal{R}_r(A) = \bigcup_i G_i, \forall i, G_i\) is r-open and \(G_i \subseteq U\). Since in a \((V)\) space (Proposition 1), the union of a family of open (r-open) sets is open (r-open). So \(\bigcup_i G_i\) is r-open. Let \(\bigcup_i G_i = G\), hence \(G\) is r-open and \(G \subseteq A\).

From the definition of \(\mathcal{R}_l(A)\), we have \(\mathcal{R}_l(A) = \bigcup_i H_i, \forall i, H_i\) is l-open and \(H_i \subseteq A\). Since in a \((V)\) space, the union of a family of open (l-open) sets is open (l-open). So \(\bigcup_i H_i\) is l-open. Let \(\bigcup_i H_i = H\), hence \(H\) is l-open and \(H \subseteq A\).

From the definition of \(\mathcal{R}_m(A)\), we have \(\mathcal{R}_m(A) = \bigcup_i M_i, \forall i, M_i\) is m-open and \(M_i \subseteq A\). Since \(G\) is r-open, it is m-open (Lemma 2) and since \(H\) is l-open, it is m-open (Lemma 2). Hence \(G \cup H\) is m-open. Therefore \(\mathcal{R}_r(A) \cup \mathcal{R}_l(A) = G \cup H, G \cup H\) is m-open and \(G \cup H \subseteq A\). So \(\mathcal{R}_r(A) \cup \mathcal{R}_l(A) = G \cup H \subseteq \bigcup_i M_i = \mathcal{R}_m(A)\). Hence \(\mathcal{R}_r(A) \cup \mathcal{R}_l(A) \subseteq \mathcal{R}_m(A)\). \(\square\)

(ii) From the definition of \(\overline{\mathcal{R}}_r(A)\), we have \(\overline{\mathcal{R}}_r(A) = \cap_i F_i, \forall i, F_i\) is r-closed and \(A \subseteq F_i\). Since in a \((V)\) space (Proposition 1), the intersection of a family of closed (r-closed) sets is closed (r-closed). So \(\cap_i F_i\) is r-closed. Let \(\cap_i F_i = F\). Hence \(F\) is r-closed and \(A \subseteq F\). From the definition of \(\overline{\mathcal{R}}_l(A)\), we have \(\overline{\mathcal{R}}_l(A) = \cap_i Z_i, \forall i, Z_i\) is l-closed and \(A \subseteq Z_i\). Since in a \((V)\) space (Proposition 1), the intersection of a family of closed (l-closed) sets is closed (l-closed). So \(\cap_i Z_i\) is l-closed. Let \(\cap_i Z_i = Z\). Hence \(Z\) is l-closed and \(A \subseteq Z\).

From the definition of \(\overline{\mathcal{R}}_m(A)\), we have \(\overline{\mathcal{R}}_m(A) = i \cap W_i, \forall i, W_i\) is m-closed and \(A \subseteq W_i\). Since \(F\) is r-closed and \(Z\) is l-closed, hence \(F\) and \(Z\) are m-closed sets (Lemma 1). So \(F \cap Z\) is also m-closed set and \(A \subseteq F \cap Z\). Therefore \(\overline{\mathcal{R}}_r(A) \cap \overline{\mathcal{R}}_l(A) = F \cap Z, F \cap Z\) is m-closed and \(A \subseteq F \cap Z\). So \(\overline{\mathcal{R}}_r(A) \cap \overline{\mathcal{R}}_l(A) \supseteq i \cap W_i = \overline{\mathcal{R}}_m(A)\). Hence \(\overline{\mathcal{R}}_m(A) \subseteq \overline{\mathcal{R}}_r(A) \cap \overline{\mathcal{R}}_l(A)\). \(\square\)
(iii) Let $x \in B_m(A) \Rightarrow x \in (\bar{R}_m(A) - \bar{R}_n(A)) \Rightarrow x \in \bar{R}_m(A) \land x \notin \bar{R}_n(A)$. Since $\bar{R}_m(A) \subseteq \bar{R}_r(A) \cap \bar{R}_l(A)$, and $\bar{R}_r(A) \cup \bar{R}_l(A) \subseteq \bar{R}_m(A)$, so $x \in (\bar{R}_r(A) \cap \bar{R}_l(A))$ \land x \notin (\bar{R}_r(A) \cup \bar{R}_l(A))$.

\[ \Rightarrow [x \in \bar{R}_r(A) \land x \notin \bar{R}_l(A)] \land [x \notin \bar{R}_r(A) \land x \notin \bar{R}_l(A)] \]
\[ \Rightarrow [x \in \bar{R}_r(A) \land x \notin \bar{R}_r(A)] \land [x \in \bar{R}_l(A) \land x \notin \bar{R}_l(A)] \]
\[ \Rightarrow x \in (\bar{R}_r(A) - \bar{R}_r(A)) \land x \in (\bar{R}_l(A) - \bar{R}_l(A)) \]
\[ \Rightarrow x \in B_r(A) \land x \in B_l(A) \]
\[ \Rightarrow x \in (B_r(A) \cap B_l(A)). \]

Hence $B_m(A) \subseteq B_r(A) \cap B_l(A)$. \qed

**Example 6:** Let $U = \{a, b, c, d, e\}$, 

\[ R = \{(a, a), (b, c), (b, d), (c, e), (c, d), (d, d), (d, a), (e, e), (e, b)\}. \]

So, the right neighborhood systems are given by

\[ NS_r(a) = \{\{a\}\}, NS_r(b) = \{\{c, d\}\}, NS_r(c) = \{\{e, d\}\}, \]
\[ NS_r(d) = \{\{d, e\}\}, NS_r(e) = \{\{c, b\}\}. \]

The left neighborhood systems are given by

\[ NS_l(a) = \{\{a\}\}, NS_l(b) = \{\{e\}\}, NS_l(c) = \{\{b\}\}, \]
\[ NS_l(d) = \{\{b, c\}\}, NS_l(e) = \{\{e, c\}\}. \]

The mixed neighborhood systems are given by

\[ NS_m(a) = \{\{a\}\}, \{a, d\}\}, \]
\[ NS_m(b) = \{\{c, d\}\}, \{e\}\}, \]
\[ NS_m(c) = \{\{e, d\}\}, \{b\}\}, \]
\[ NS_m(d) = \{\{d, a\}\}, \{d, b, c\}\}, \]
\[ NS_m(e) = \{\{e, b\}\}, \{c, e\}\}. \]
Generalized Topological Approach for Rough Sets

The sets $A'_t$, $A'_j$, and $A'_m$ for all subsets of $U$ are given in the following Table.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$A'_t$</th>
<th>$A'_j$</th>
<th>$A'_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a}$</td>
<td>${d}$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${b}$</td>
<td>${e}$</td>
<td>${c, d}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${c}$</td>
<td>${b}$</td>
<td>${d, e}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${d}$</td>
<td>${b, c}$</td>
<td>${a}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${e}$</td>
<td>${c}$</td>
<td>${b}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${a, b}$</td>
<td>${d, e}$</td>
<td>${c, d}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>${a, c}$</td>
<td>${b, d}$</td>
<td>${d, e}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>${a, d}$</td>
<td>${b, c, d}$</td>
<td>${a}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${a, e}$</td>
<td>${c, d}$</td>
<td>${b}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${b, c}$</td>
<td>${b, e}$</td>
<td>${c, d, e}$</td>
<td>${e}$</td>
</tr>
<tr>
<td>${b, d}$</td>
<td>${b, c, e}$</td>
<td>${a, c, d}$</td>
<td>${c}$</td>
</tr>
<tr>
<td>${b, e}$</td>
<td>${c, e}$</td>
<td>${b, c, d}$</td>
<td>${c}$</td>
</tr>
<tr>
<td>${c, d}$</td>
<td>${b, c}$</td>
<td>${a, d, e}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${c, e}$</td>
<td>${b, c}$</td>
<td>${b, d, e}$</td>
<td>${b}$</td>
</tr>
<tr>
<td>${d, e}$</td>
<td>${b, c}$</td>
<td>${a, b}$</td>
<td>${b}$</td>
</tr>
<tr>
<td>${a, b, c}$</td>
<td>${b, d, e}$</td>
<td>${c, d, e}$</td>
<td>${d, e}$</td>
</tr>
<tr>
<td>${a, b, d}$</td>
<td>${b, c, d, e}$</td>
<td>${a, c, d}$</td>
<td>${c, d}$</td>
</tr>
<tr>
<td>${a, b, e}$</td>
<td>${c, d, e}$</td>
<td>${b, c, d}$</td>
<td>${c, d}$</td>
</tr>
<tr>
<td>${a, c, d}$</td>
<td>${b, c, d}$</td>
<td>${a, d, e}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>${a, c, e}$</td>
<td>${b, c, d}$</td>
<td>${b, d, e}$</td>
<td>${b, d}$</td>
</tr>
<tr>
<td>${a, d, e}$</td>
<td>${b, c, d}$</td>
<td>${a, b}$</td>
<td>${b}$</td>
</tr>
<tr>
<td>${b, c, d}$</td>
<td>${b, c, e}$</td>
<td>${a, c, d, e}$</td>
<td>${c, e}$</td>
</tr>
<tr>
<td>${b, c, e}$</td>
<td>${b, c, d, e}$</td>
<td>${b, c, d, e}$</td>
<td>${b, c, e}$</td>
</tr>
<tr>
<td>${b, d, e}$</td>
<td>${b, c, e}$</td>
<td>${a, b, c, d}$</td>
<td>${b, c}$</td>
</tr>
<tr>
<td>${c, d, e}$</td>
<td>${b, c}$</td>
<td>${a, b, d, e}$</td>
<td>${b}$</td>
</tr>
<tr>
<td>${a, b, c, d}$</td>
<td>${b, c, d, e}$</td>
<td>${a, c, d, e}$</td>
<td>${c, d, e}$</td>
</tr>
<tr>
<td>${a, b, c, e}$</td>
<td>${b, c, d, e}$</td>
<td>${b, c, d, e}$</td>
<td>${b, c, e}$</td>
</tr>
<tr>
<td>${a, b, d, e}$</td>
<td>${b, c, d, e}$</td>
<td>${a, b, c, d}$</td>
<td>${b, c, d}$</td>
</tr>
<tr>
<td>${a, c, d, e}$</td>
<td>${b, c, d}$</td>
<td>${a, b, d, e}$</td>
<td>${b, d}$</td>
</tr>
<tr>
<td>${b, c, d, e}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${b, c, d, e}$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$U$</td>
<td>${b, c, d, e}$</td>
<td>$U$</td>
<td>${b, c, d, e}$</td>
</tr>
</tbody>
</table>
From Definition 12 and the above table we note that $A'_m = A'_r \cap A'_l$.

From Definitions 14, 15 and the above table we have

\[ \Gamma_r = \{ U, \phi, \{ b, c, e \}, \{ b, c, d, e \} \} \]
\[ \Gamma_l = \{ U, \phi, \{ a \}, \{ a, d \} \}, \text{ and} \]
\[ \Gamma_m = \{ U, \phi, \{ a \}, \{ b \}, \{ c \}, \{ d \}, \{ e \}, \{ a, d \}, \{ a, e \}, \{ c, d \}, \{ a, c, d \}, \]
\[ \{ b, c, e \}, \{ b, c, d, e \}. \]

We also have

\[ \Omega_r = \{ U, \phi, \{ a \}, \{ a, d \} \}, \]
\[ \Omega_l = \{ U, \phi, \{ b, c, e \}, \{ b, c, d, e \} \}, \text{ and} \]
\[ \Omega_m = \{ U, \phi, \{ a \}, \{ a, d \}, \{ b \}, \{ e \}, \{ a, b \}, \{ c \}, \{ e \}, \{ b, c, e \}, \{ b, c, d, e \}, \]
\[ \{ a, b, c, d \}, \{ a, b, d, e \}, \{ a, c, d, e \} \}. \]

The lower approximations of some subsets of $V$ using (right, left, and mixed) neighborhood systems are tabulated in the following Table.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$R_r(A)$</th>
<th>$R_l(A)$</th>
<th>$R_m(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ a }$</td>
<td>${ a }$</td>
<td>$\phi$</td>
<td>${ a }$</td>
</tr>
<tr>
<td>${ c }$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${ d }$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${ a, d }$</td>
<td>${ a, d }$</td>
<td>$\phi$</td>
<td>${ a, d }$</td>
</tr>
<tr>
<td>${ b, e }$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>${ b, e }$</td>
</tr>
<tr>
<td>${ b, c, e }$</td>
<td>$\phi$</td>
<td>${ b, c, e }$</td>
<td>${ b, c, e }$</td>
</tr>
<tr>
<td>${ a, c, e }$</td>
<td>${ a }$</td>
<td>$\phi$</td>
<td>${ a }$</td>
</tr>
<tr>
<td>${ b, c, d }$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>${ b, c, d }$</td>
</tr>
<tr>
<td>${ a, b, d, e }$</td>
<td>${ a, d }$</td>
<td>$\phi$</td>
<td>${ a, b, d, e }$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
</tbody>
</table>

We note from the above table that $R_r(A) \cup R_l(A) \subseteq R_m(A)$. The upper approximations of some subsets of $U$ using (right, left, and mixed) neighborhood systems are tabulated in the following Table.
We note from the above table that $\bar{R}(A) \subseteq \bar{R}(A) \cap \bar{R}(A)$. The boundary sets of the same subsets of $U$ using (right, left, and mixed) neighborhood systems are tabulated in the following Table.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\bar{R}(A)$</th>
<th>$\bar{R}(A)$</th>
<th>$\bar{R}(m)(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a}$</td>
<td>$U$</td>
<td>${a}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>${c}$</td>
<td>$\phi$</td>
<td>$U$</td>
<td>${c}$</td>
</tr>
<tr>
<td>${d}$</td>
<td>$\phi$</td>
<td>${a, d}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>${a, d}$</td>
<td>$U$</td>
<td>${a, d}$</td>
<td>${a, d}$</td>
</tr>
<tr>
<td>${b, e}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${b, c, e}$</td>
</tr>
<tr>
<td>${b, c, e}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${b, c, e}$</td>
</tr>
<tr>
<td>${a, c, e}$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
<td>${b, c, d}$</td>
<td>${b, c, d, e}$</td>
<td>$U$</td>
<td>${b, c, d, e}$</td>
</tr>
<tr>
<td>${a, b, d, e}$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
</tbody>
</table>

We note from the above table that $B_{m}(A) \subseteq B_{r}(A) \cap B_{l}(A)$. The boundary sets of the same subsets of $U$ using (right, left, and mixed) neighborhood systems are tabulated in the following Table.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B_{r}(A)$</th>
<th>$B_{l}(A)$</th>
<th>$B_{m}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a}$</td>
<td>${b, c, d, e}$</td>
<td>${a}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${c}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${c}$</td>
</tr>
<tr>
<td>${d}$</td>
<td>${b, c, d, e}$</td>
<td>${a, d}$</td>
<td>${d}$</td>
</tr>
<tr>
<td>${a, d}$</td>
<td>${b, c, e}$</td>
<td>${a, d}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${b, e}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${c}$</td>
</tr>
<tr>
<td>${b, c, e}$</td>
<td>${b, c, e}$</td>
<td>${a, d}$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>${a, c, e}$</td>
<td>${b, c, d, e}$</td>
<td>$U$</td>
<td>${b, c, d, e}$</td>
</tr>
<tr>
<td>${b, c, d}$</td>
<td>${b, c, d, e}$</td>
<td>$U$</td>
<td>${e}$</td>
</tr>
<tr>
<td>${a, b, d, e}$</td>
<td>${b, c, e}$</td>
<td>$U$</td>
<td>${c}$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
<tr>
<td>$U$</td>
<td>$\phi$</td>
<td>$\phi$</td>
<td>$\phi$</td>
</tr>
</tbody>
</table>

We note from the above table that $B_{m}(A) \subseteq B_{r}(A) \cap B_{l}(A)$.
5. PROPERTIES OF THE APPROXIMATIONS BASED ON MIXED NEIGHBORHOOD SYSTEMS

Proposition 2: Let \((U, R)\) be a MNAS and \(A, B \subseteq U\), then:

1. \(R_m(A) \subseteq (A) \subseteq \bar{R}_m(A)\),
2. \(R_m(U) = U = \bar{R}_m(U), R_m(\phi) = \phi = \bar{R}_m(\phi)\),
3. \(A \subseteq B \Rightarrow R_m(A) \subseteq \bar{R}_m(B)\),
4. \(A \subseteq B \Rightarrow \bar{R}_m(A) \subseteq \bar{R}_m(B)\).

Proof: 1. \(R_m(A) = \bigcup_i G_i, G_i\) is m-open and \(\forall i, G_i \subseteq A\). Which implies \(\bigcup_i G_i \subseteq A \Rightarrow R_m(A) \subseteq A\).

\[ \bar{R}_m(A) = \bigcap_i F_i, F_i\) is m-closed and \(\forall i, A \subseteq F_i\). Which implies \(A \subseteq \bigcap_i F_i \Rightarrow A \subseteq \bar{R}_m(A)\).

2. Obvious, since \(U\) and \(\phi\) are both m-closed and m-open.

3. \(R_m(A) \subseteq A \subseteq B \Rightarrow R_m(A) \subseteq B\). But \(R_m(A)\) is open, since the union of a family of m-open sets is m-open. Hence \(R_m(A) \subseteq \bar{R}_m(B)\). Because \(R_m(B)\) is the union of all m-open sets contained in \(B\).

4. \(A \subseteq B \subseteq \bar{R}_m(B) \Rightarrow A \subseteq \bar{R}_m(B)\). But \(\bar{R}_m(B)\) is m-closed, since the intersection of a family of m-closed sets is closed. Therefore \(\bar{R}_m(A) \subseteq \bar{R}_m(B)\), because \(\bar{R}_m(A)\) is the intersection of all m-closed sets containing \(A\).

Proposition 3: Let \((U, R)\) be a MNAS and \(A, B \subseteq U\), then:

1. \(R_m(A \cap B) \subseteq R_m(A) \cap \bar{R}_m(B)\),
2. \(R_m(A) \cup R_m(B) \subseteq R_m(A \cup B)\),
3. \(\bar{R}_m(A \cap B) \subseteq \bar{R}_m(A) \cap \bar{R}_m(B)\),
4. \(\bar{R}_m(A) \cup \bar{R}_m(B) \subseteq \bar{R}_m(A \cup B)\).

Proof: Obvious.

Proposition 3: Let \((U, R)\) be a MNAS and \(A, B \subseteq U\), then:

1. \(R_m(U - A) = U - \bar{R}_m(A)\),
2. \(\bar{R}_m(U - A) = U - R_m(A)\),
3. \(R_m(R_m(A)) = R_m(A)\),
4. \(\bar{R}_m(\bar{R}_m(A)) = \bar{R}_m(A)\).
Proof: 1. \( U - \overline{R}_m(A) = U - \bigcap_i F_i \), \( F_i \) is \( m \)-closed and \( \forall i, A \subseteq F_i \).

\[
\therefore \ U - \overline{R}_m(A) = \bigcup_i (U - F_i). \\
\text{Put } G_i = U - F_i \text{ then } G_i \text{ is } m \text{-open and } \\
\forall i, G_i \subseteq U - A.
\]

\[
\therefore \ U - \overline{R}_m(A) = \bigcup_i G_i = \overline{R}_m(U - A). \\ 
\text{Hence } \overline{R}_m(U - A) = U - \overline{R}_m(A).
\]

2. \( \overline{R}_m(U - A) = \bigcap_i F_i \), \( F_i \) is \( m \)-closed and \( \forall i, U - A \subseteq F_i \). Hence \( U - F_i \subseteq A \). Put \( G_i \subseteq U - F_i \), then \( G_i \) is \( m \)-open and \( \forall i, G_i \subseteq A. 
\]

\[
\therefore \ U - \overline{R}_m(U - A) = U - \bigcap_i F_i = \bigcup_i (U - F_i) = \bigcup_i G_i = \overline{R}_m(A), \\
\therefore \ \overline{R}_m(U - A) = U - \overline{R}_m(A).
\]

3. Since \( A \) is \( m \)-open iff \( A = \overline{R}_m(A) \) and \( \overline{R}_m(A) \) is \( m \)-open (since it is a union of a family of \( m \)-open sets), then \( \overline{R}_m(\overline{R}_m(A)) = \overline{R}_m(A). \)

4. Since \( A \) is \( m \)-closed iff \( A = \overline{R}_m(A) \) and \( \overline{R}_m(A) \) is \( m \)-closed (since it is an intersection of a family of \( m \)-closed sets), then \( \overline{R}_m(\overline{R}_m(A)) = \overline{R}_m(A). \)

Remark 3: Let \((U, R)\) be a MNAS, then the following properties are not true in general for every \( A, B \subseteq U: \)

1. \( \overline{R}_m(A \cup B) \subseteq \overline{R}_m(A) \cup \overline{R}_m(B), \)
2. \( \overline{R}_m(A) \cap \overline{R}_m(B) \subseteq \overline{R}_m(A \cap B), \)
3. \( \overline{R}_m(\overline{R}_m(A)) = \overline{R}_m(A), \)
4. \( \overline{R}_m(\overline{R}_m(A)) = \overline{R}_m(A). \)

The following example show Remark 3

Example 7: Consider example 6 with mixed neighborhoods as a counter example, we have

1. Let \( A = \{a\}, B = \{b\}, \) then \( \overline{R}_m(A) = \{a\}, \overline{R}_m(B) = \{b\}, \) and \( \overline{R}_m(A \cup B) = \overline{R}_m(a, b) = U. \)
   Obviously, \( \overline{R}_m(A \cup B) \not\subseteq \overline{R}_m(A) \cup \overline{R}_m(B). \)
2. Let \( A = \{a, b, c, d\}, B = \{a, b, d, e\}, \) then \( \overline{R}_m(A) = \{a, b, c, d\}, \overline{R}_m(B) = \{a, b, d, e\}, \)
   and \( \overline{R}_m(A \cap B) = \overline{R}_m(a, b, d) = \{a, d\}, \overline{R}_m(A) \cap \overline{R}_m(B) = \{a, b, d\}. \)
   Obviously, \( \overline{R}_m(A) \cap \overline{R}_m(B) \not\subseteq \overline{R}_m(A \cap B). \)
3. Let \( A = \{a, b, d, e\}, \) then \( \overline{R}_m(A) = \{a, b, d, e\}, \overline{R}_m(\overline{R}_m(A)) = \overline{R}_m(a, b, d, e) = U. \)
   Obviously, \( \overline{R}_m(\overline{R}_m(A)) \neq \overline{R}_m(A). \)
4. Let $A = \{c\}$, then $\bar{R}_m(A) = \{c\}$, $R_m(\bar{R}_m(A)) = R_m(\{c\}) = \phi$. Therefore $R_m(\bar{R}_m(A)) \neq \bar{R}_m(A)$.

6. CONCLUSION

This paper (Theorem 1) proves that approximations based on mixed neighborhood systems are accurate than approximations based on either right neighborhood systems or left neighborhood systems or using both of them (by using both of them, we mean taking the intersection $\bar{R}(A) \cap \bar{R}(A)$ to be an upper approximation of $A$ and the union $\bar{R}(A) \cup \bar{R}(A)$ to be a lower approximation of).

REFERENCES


Generalized Topological Approach for Rough Sets


M. E. Abd El-Monsef, A. S. Salama and O. A. Embaby
Department of Mathematics, Faculty of Science,
Tanta University, Egypt
E-mail: (monsef@dr.com), (dr_salama75@yahoo.com),
and (Oembaby@yahoo.com)