A Comprehensive Study of Rough Sets and Rough Fuzzy Sets on Two Universes

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Abstract—In rough set theory, the lower and upper approximation operators can be constructed via a variety of approaches. Various generalizations of rough approximation operators have been made over the years. This paper presents a framework for the study of rough sets and rough fuzzy sets on two universes of discourse. By means of a binary relation between two universes of discourse, a class of revised rough sets and revised rough fuzzy sets based on two universes have been proposed. Some properties of the new model are revealed. We believe that this model will be more natural in the sense that rough sets (resp. rough fuzzy sets) are approximated by sets (resp. fuzzy sets) on the same universe. Moreover, some results, examples and counter examples are provided.

Index Terms—approximation operator; generalized rough set; generalized rough fuzzy set; inverse serial relation; revised rough set; revised rough fuzzy set; strong inverse serial relation.

1 INTRODUCTION

ROUGH set theory was developed by Pawlak as a formal tool for representing and processing information in database. In Pawlak rough set theory [14], [15], the lower and upper approximation operators are based on equivalence relation. However, the requirement of an equivalence relation in Pawlak rough set models seems to be a very restrictive condition that may limit the applications of the rough set models. Thus one of the main directions of research in rough set theory is naturally the generalization of the Pawlak rough set approximations. For instance, the notations of approximations are extended to general binary relations [1], [5], [6], [17], [21], [27], [29], [31], [35], neighborhood systems [8], [30], coverings [36], completely distributive lattices [2], fuzzy lattices [13] and Boolean algebras [12], [18]. On the other hand, the generalization of rough sets in fuzzy environment is another topic receiving much attention in recent years [4], [8], [9], [10], [11], [16], [19]. Based on equivalence relation, the concepts of fuzzy rough and rough fuzzy set were first proposed by Dubois and Prade [4] in the Pawlak approximation space. Yao [28] gave a unified model for both rough fuzzy sets and fuzzy rough sets based on the analysis of level sets of fuzzy sets. Pei [19], [20] and Cock, et al.,[3] considered the approximation problems of fuzzy sets in fuzzy information systems result in theory of fuzzy rough sets. Li and Zhang [7] analyzed crisp binary relations and rough fuzzy approximations.

Rough set models can also be extended via two universes of discourse [24], [25], [26], [27], [33]. In [22], [23], [34], they were interpreted as interval structure. Though these models have different methods of computation, they start with almost the “same” framework (two universes of discourse with a relation). It should also be noted that the approximated sets and the approximating sets in these models always located at two different universes of discourse. This is, however, not natural and is inconvenient for knowledge discovery by means of rough set theory. This issue was seldom investigated in the literature. As an exception, Pei and Xu [20] proposed such kind of rough set models on two universes of discourse. The objective of this paper is to establish another kind of rough set models via a common structure with a relation between two universes of discourse, and to extend the rough set models to the corresponding fuzzy environments. It should be noted that these rough set models can guarantee that the approximating sets and the approximated sets are on the same universe of discourse.

2 Preliminaries

Let $U$ be a finite and nonempty set called the universe. The class of all crisp subsets (respectively, fuzzy subsets) of $U$ will be denoted by $P(U)$ (respectively, by $F(U)$).

Let $U$ and $V$ be two finite and nonempty universes and let $R$ be a binary relation from $U$ to $V$, the triple $(U,V,R)$ is called (two-universe) approximation space. Then the relation $R$ is called:

1. Serial if for all $x \in U$, there exists $y \in V$ such that $(x,y) \in R$.
2. Inverse serial if for all $y \in V$, there exists $x \in U$ such that $(x,y) \in R$.
3. Compatibility relation, if $R$ is both serial and inverse serial.

If $U = V$, $R$ is referred to as a binary relation on $U$ and is
Definition 2.1 [33]. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space. Then, a set-valued mapping \(F\) from \(U\) to \(\mathcal{P}(V)\) representing the successor neighborhood of \(x\) with respect to \(R\), as follows:

\[
F : U \to \mathcal{P}(V), \quad F(x) = \{ y \in V : (x,y) \in R \}.
\]

Definition 2.2 [33]. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space. Then, the lower and upper approximations of \(Y \in \mathcal{P}(V)\) are respectively defined as follows:

\[
\begin{align*}
\hat{R}(Y) &= \{ x \in U : F(x) \subseteq Y \}, \\
\overline{R}(Y) &= \{ x \in U : F(x) \cap Y \neq \emptyset \}.
\end{align*}
\]

The ordered set-pair \((\hat{R}(Y),\overline{R}(Y))\) is called a generalized rough set. A subset \(Y \in \mathcal{P}(V)\) is called definable or exact with respect to \((U,V,R)\) if \(\hat{R}(Y) = \overline{R}(Y)\), otherwise it is undefinable, or rough.

Proposition 2.1 [33]. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space and let \(R\) be a compatibility relation. Then, for all \(Y_1,Y_2 \in \mathcal{P}(V)\), the approximation operators satisfy the following properties:

\[
\begin{align*}
(L_1) \quad \overline{R}(Y) &= (\hat{R}(Y))^c, \text{ where } Y^c \text{ denotes the complement of } Y \text{ in } V, \\
(L_2) \quad \overline{R}(U) &= U, \\
(L_3) \quad \overline{R}(Y_1 \cap Y_2) &= \overline{R}(Y_1) \cap \overline{R}(Y_2), \\
(L_4) \quad \overline{R}(Y_1 \cup Y_2) &= \overline{R}(Y_1) \cup \overline{R}(Y_2), \\
(L_5) \quad Y_1 \subseteq Y_2 \implies \overline{R}(Y_1) \subseteq \overline{R}(Y_2), \\
(L_6) \quad \overline{R}(\emptyset) &= \emptyset, \\
(U_1) \quad \hat{R}(Y) &= (\overline{R}(Y))^c, \\
(U_2) \quad \hat{R}(U) &= \emptyset, \\
(U_3) \quad \hat{R}(Y_1 \cap Y_2) &= \hat{R}(Y_1) \cap \hat{R}(Y_2), \\
(U_4) \quad \hat{R}(Y_1 \cup Y_2) &= \hat{R}(Y_1) \cup \hat{R}(Y_2), \\
(U_5) \quad Y_1 \subseteq Y_2 \implies \hat{R}(Y_1) \subseteq \hat{R}(Y_2), \\
(U_6) \quad \hat{R}(U) &= \emptyset, \\
(U_7) \quad \hat{R}(Y) \subseteq \overline{R}(Y).
\end{align*}
\]

Properties \((L_1)\) and \((U_7)\) show that the approximation operators \(R\) and \(\overline{R}\) are dual to each other. Properties with the same number may be considered as dual properties. With respect to certain special types, say, reflexive, symmetric, transitive, and Euclidean binary relations on the universe \(U\), the approximation operators have additional properties [29], [30], [32].

Proposition 2.2. Let \(R \in \mathcal{P}(U \times U)\) be an arbitrary binary relation on \(U\). Then, \(\forall X \in \mathcal{P}(U)\) \(R\) is reflexive if \(\forall x, y \in U\), \((x,x) \in R\) implies that \((y,y) \in R\). \(R\) is symmetric if \(\forall x, y \in U\), \((x,y) \in R\) implies that \((y,x) \in R\). \(R\) is transitive if \(\forall x, y, z \in U\), \((x,y) \in R\) and \((y,z) \in R\) imply that \((x,z) \in R\). \(R\) is Euclidean if \(\forall x, y, z \in U\), \((x,y) \in R\) and \((y,z) \in R\) imply that \((x,z) \in R\).

3 Revised Generalized Rough Sets

Because reflexivity, symmetry and transitivity are meaningless for binary relations from \(U\) to \(V\), the properties \((L_1) - (L_10)\) and \((U_1) - (U_10)\) which are true in various generalized rough set models do not hold in two-universe models. However, in the above model for generalized rough sets, subsets of the universe \(V\) are approximated by subsets of the other universe \(U\). This seems very unreasonable. Thus a more natural form for rough sets on two universes is proposed such that the approximations of subsets of the universe \(V\) are subsets of the universe \(U\). Therefore we will modify these models in this section.

Definition 3.1. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space. Then we can define a revised mapping \(G\) from \(V\) to \(\mathcal{P}(V)\) induced by \(R\) as follows:

\[
G : V \rightarrow \mathcal{P}(V), \quad G(y) = \begin{cases} 
\{ x \in U : y \in F(x) \}, & \text{if } y \in F(x) \subseteq (\bigcap F(x)) \cap \emptyset, \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

Definition 3.2. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space, an inverse serial relation \(R \in \mathcal{P}(U \times V)\) is called strong inverse serial if for all \(y_1,y_2 \in V\), \(G(y_1) \cap G(y_2) \neq \emptyset\) implies that \(G(y_1) = G(y_2)\).

Lemma 3.1. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space. Then \(\forall y_1,y_2 \in V\) if \(y_1 \in G(y_2)\), then \(G(y_1) \subseteq G(y_2)\).

Proof. Let \(y \in V\) such that \(y \notin G(y_2)\), then there exists \(x_1 \in U\) such that \(y \subseteq F(x_1)\) and \(y \notin F(x_1)\). But \(y \in G(y_2)\) this implies that \(y \subseteq F(x_1)\), i.e., \(G(y_1) \subseteq F(x_1)\), thus \(y \notin G(y_1)\) and hence \(G(y_1) \subseteq G(y_2)\).

Lemma 3.2. Let \((U,V,R)\) be a \((\text{two-universe})\) approximation space, if \(R\) is inverse serial, then, \(y \in G(y) \), \(\forall y \in V\).

Proof. Since \(R\) is inverse serial, then \(G(y) \neq \emptyset \) \(\forall y \in V\) and hence \(y \in G(y)\), \(\forall y \in V\).

Definition 3.3. Let \(U\) and \(V\) be two nonempty finite universes, and \(R \in \mathcal{P}(U \times V)\) a binary relation from \(U\) to \(V\). The ordered triple \((U,V,R)\) is called a revised two universe approximation space. The revised lower and upper approximations of \(Y \in \mathcal{P}(V)\) are defined respectively as follows:
\[ R'(Y) = \{ y \in V \mid G(y) \subseteq Y \}, \]
\[ \overline{R}(Y) = \{ y \in V \mid G(y) \cap Y \neq \emptyset \}. \]

**Proposition 3.1.** Let \((U, V, R)\) be a revised (two-universe) approximation space. Then for all \(Y, Y_1, Y_2 \in \mathcal{P}(V)\), the approximation operators have the following properties:

1. \[ L'_{10} \quad \overline{R}(Y) \subseteq \overline{R}(R(Y)). \]
2. \[ L'_{10} \quad \overline{R}(Y) \subseteq \overline{R}(R(Y)). \]
3. \[ U'_{10} \quad \overline{R}(R(Y)) \subseteq R(Y). \]
4. \[ LU' \quad \overline{R}(R(Y)) \subseteq R(Y). \]

The following example illustrates Remark 3.1.

**Example 3.1.** Let \( U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \), \( V = \{y_1, y_2, y_3, y_4, y_5, y_6\} \) and \( R \in \mathcal{P}(U \times V) \) be a binary relation defined as:

<table>
<thead>
<tr>
<th>( R )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
<th>( y_4 )</th>
<th>( y_5 )</th>
<th>( y_6 )</th>
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<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>( x_2 )</td>
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<td>( x_3 )</td>
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<tr>
<td>( x_4 )</td>
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<td>0</td>
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<td>( x_5 )</td>
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<td>0</td>
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<td>1</td>
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</tr>
<tr>
<td>( x_6 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>( x_7 )</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence we have \( R'(\emptyset) = \{y_2\} \neq \emptyset \) and \( \overline{R}(V) = \{y_1, y_2, y_3, y_4, y_5, y_6\} \), i.e., \( L'_{5} \) and \( U'_4 \) do not hold.

**Proposition 3.2.** Let \((U, V, R)\) be a revised (two-universe) approximation space and \( R \) be an inverse serial relation. Then for all \( Y \in \mathcal{P}(V)\), the approximation operators have the following properties:

1. \[ L''_{6} \quad R'(\emptyset) = \emptyset. \]
2. \[ L''_{7} \quad \overline{R}(Y) \subseteq Y. \]
3. \[ U''_{6} \quad \overline{R}(V) = V. \]
4. \[ U''_{7} \quad Y \subseteq \overline{R}(Y). \]
5. \[ LU'' \quad \overline{R}(Y) \subseteq R(Y). \]

**Proof.** By the duality of approximation operators, we only need to prove the properties \( L''_{6}, L''_{7}, \) and \( LU'' \).

**Remark 3.1.** If \( R \in \mathcal{P}(U \times V)\) is a binary relation in a revised (two-universe) approximation space \((U, V, R)\), then the following properties do not hold for all \( Y \in \mathcal{P}(V)\):

1. \[ L''_{6} \quad R'(\emptyset) = \emptyset. \]
2. \[ L''_{7} \quad \overline{R}(Y) \subseteq Y. \]
Remark 3.2. If $R \in \mathcal{P}(U \times V)$ is an inverse serial relation in a revised (two-universe) approximation space $(U, V, R)$, then the following properties do not hold for all $Y \in \mathcal{P}(V)$:

$$
L^*_{0.8} Y \subseteq R^*(Y) .
$$

$$
L^*_{1.0} \overline{R}(Y) \subseteq R^*(Y) .
$$

$$
U^*_{0.8} \overline{R}(Y) \subseteq Y .
$$

$$
U^*_{1.0} \overline{R}(Y) \subseteq R^*(Y) .
$$

The following example illustrates Remark 3.2.

Example 3.2. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $V = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $R \in \mathcal{P}(U \times V)$ be a binary relation defined as

<table>
<thead>
<tr>
<th>$R$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
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<tr>
<td>$x_1$</td>
<td>0</td>
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<td>$x_2$</td>
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<td>$x_3$</td>
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<td>$x_4$</td>
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<td>$x_5$</td>
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<td>$x_6$</td>
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<td>$x_7$</td>
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</tr>
</tbody>
</table>

If $Y = \{y_2, y_3, y_4, y_5, y_6\}$, then $R^*(Y) = \{y_2, y_3, y_4\}$ and $\overline{R}(Y) = \{y_2, y_3, y_4, y_5, y_6\}$, hence $\overline{R}(R^*(Y)) \subseteq Y$ and $\overline{R}(\overline{R}(Y)) \subseteq \overline{R}(Y)$, i.e., $U^*_{0.8}$ and $U^*_{1.0}$ do not hold.

Proposition 3.3. Let $(U, V, R)$ be a revised (two-universe) approximation space and $R$ be a strong inverse serial relation. Then for all $Y \in \mathcal{P}(V)$, the approximation operators have the following properties:

$$
L^*_{0.8} Y \subseteq R^*(Y) .
$$

$$
L^*_{1.0} \overline{R}(Y) \subseteq R^*(Y) .
$$

$$
U^*_{0.8} \overline{R}(Y) \subseteq Y .
$$

$$
U^*_{1.0} \overline{R}(Y) \subseteq R^*(Y) .
$$

Proof. $L^*_{1.0}$. Assume that $y \in \overline{R}(Y)$, then $G(y) \subseteq \overline{R}(Y)$, i.e., there exists $x \in V$ such that $x \in G(y)$ and $x \in \overline{R}(Y)$, this implies that $G(x) \cap Y = \emptyset$. Since $R$ is strong inverse serial, then $G(x) = G(y)$. Consequently $G(y) \cap Y = \emptyset$, i.e., $y \notin \overline{R}(Y)$. Therefore $\overline{R}(Y) \subseteq \overline{R}(Y)$.

$L^*_{0.8}$. The proof of $L^*_{0.8}$ comes from $U^*_{0.7}$ and $L^*_{1.0}$.

In the same manner we can prove $U^*_{0.8}$ and $U^*_{1.0}$.

4 Generalized Rough Fuzzy Sets

A rough fuzzy set is a generalization of rough set, derived from the approximation of fuzzy set in a crisp approximation space. In this section, we will introduce another generalization of rough fuzzy set based on the intersection of right neighborhoods.

Definition 4.1[26]. Let $U$ and $V$ be two finite non-empty universes of discourse and $R \in \mathcal{P}(U \times V)$ a binary relation from $U$ to $V$. The ordered triple $(U, V, R)$ is called a (two-universe) approximation space. For any fuzzy set $Y \in \mathcal{F}(V)$, the lower and upper approximations of $Y$, $\overline{R}(Y)$ and $\overline{R}(Y)$, with respect to the approximation space are fuzzy sets of $U$ whose membership functions, for each $x \in U$, are defined, respectively, by

$$
\overline{R}(Y)(x) = \min\{Y(y) \mid y \in F(x)\}
$$

$$
\overline{R}(Y)(x) = \max\{Y(y) \mid y \in F(x)\},
$$

where $F(x)$ is the successor neighborhood of $x$ defined in Definition 2.1.

The ordered set-pair $(\overline{R}(Y), \overline{R}(Y))$ is referred to as a generalized rough fuzzy set, and $\overline{R}(Y)$ and $\overline{R}(Y) : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ are referred to as lower and upper generalized rough fuzzy approximation operators, respectively.

Proposition 4.1[26]. In a (two-universe) model $(U, V, R)$ with compatibility relation $R$, the approximation operators satisfy the following properties for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:

$$
L^*_1 \overline{R}(Y) = \left( \overline{R}(Y^c) \right)^c , \text{ where } Y^c \text{ denotes the complement of the fuzzy subset } Y \text{ in } V .
$$

$$
L^*_2 \overline{R}(V) = U .
$$

$$
L^*_3 \overline{R}(Y \cap Y_2) = \overline{R}(Y) \cap \overline{R}(Y_2) .
$$

$$
L^*_4 \overline{R}(Y_1 \cup Y_2) \supseteq \overline{R}(Y_1) \cup \overline{R}(Y_2) .
$$

$$
L^*_5 \overline{R}(Y_1) \subseteq \overline{R}(Y_2) \implies \overline{R}(Y_1) \subseteq \overline{R}(Y_2) .
$$

$$
L^*_6 \overline{R}(\emptyset) = \emptyset .
$$

$$
L^*_7 \overline{R}(Y) = \left( \overline{R}(Y^c) \right)^c .
$$

$$
L^*_8 \overline{R}(\emptyset) = \emptyset .
$$

$$
L^*_9 \overline{R}(Y_1) \cup \overline{R}(Y_2) = \overline{R}(Y_1) \cup \overline{R}(Y_2) .
$$

$$
L^*_10 \overline{R}(Y_1) \cap \overline{R}(Y_2) \subseteq \overline{R}(Y_1) \cap \overline{R}(Y_2) .
$$

$$
L^*_11 \overline{R}(Y) = U .
$$

$$
L^*_12 \overline{R}(U) = \overline{R}(Y) .
$$

Proposition 4.2[26]. Let $R \in \mathcal{P}(U \times U)$ be an arbitrary binary relation on $U$. Then $\forall X \in \mathcal{F}(U)$,

1. $R$ is reflexive $\iff (L^*_0) R(X) \subseteq X$, $\iff (L^*_0) X \subseteq \overline{R}(X)$.

2. $R$ is symmetric $\iff (L^*_0) \overline{R}(R(X)) \subseteq X$, $\iff (U^*_0) X \subseteq \overline{R}(R(X))$.

3. $R$ is transitive $\iff (L^*_0) R(X) \subseteq R(R(X))$, $\iff (U^*_0) \overline{R}(R(X)) \subseteq R(X)$.

4. $R$ is Euclidean $\iff (L^*_0) \overline{R}(R(X)) \subseteq R(X)$, $\iff (U^*_0) \overline{R}(R(X)) \subseteq R(X)$. 

5 Revised Generalized Rough Fuzzy Sets

In the above model for generalized rough fuzzy sets, fuzzy subsets of the universe U are approximated by fuzzy subsets of the other universe V. This seems very unreasonable. Furthermore, there exists no relation between the set and its lower and upper approximations. However, the operators $R$, $\bar{R}$, $R\bar{R}$, and $R\bar{R}$ are not defined, so the properties (L1) - (L10) and (U1) - (U10) which are true in various generalized rough set models do not hold in (two-universe) models. Thus a more natural form for rough sets on two universes is proposed such that the approximations of subsets of the universe V are subsets of the universe V.

Definition 5.1. Let $(U, V, R)$ be a revised (two-universe) approximation space. Then the revised lower and upper approximations of $Y \in \mathcal{F}(V)$ are defined respectively as follows:

\[
R^*(Y)(y) = \min\{Y(z) \mid z \in G(y)\}
\]

\[
\bar{R}^*(Y)(y) = \max\{Y(z) \mid z \in G(y)\}.
\]

The pair $(R^*(x), \bar{R}^*(x))$ is referred to as a revised rough fuzzy set, and $R^*$ and $\bar{R}^*$ : $\mathcal{F}(V) \to \mathcal{F}(V)$ are referred to as the lower and upper revised rough fuzzy approximation operators, respectively.

Proposition 5.1. In a revised (two-universe) approximation space $(U, V, R)$, the approximation operators have the following properties for all $Y, Y_1, Y_2 \in \mathcal{F}(V)$:

$L^*_1$. $R^*(Y) = (\bar{R}^*(Y))^c$.

$L^*_2$. $\bar{R}^*(V) = V$.

$L^*_3$. $R^*(Y_1 \cap Y_2) = R^*(Y_1) \cap \bar{R}^*(Y_2)$.

$L^*_4$. $R^*(Y_1 \cup Y_2) \supseteq \bar{R}^*(Y_1) \cup \bar{R}^*(Y_2)$.

$L^*_5$. $Y_1 \subseteq Y_2 \Rightarrow \bar{R}^*(Y_1) \subseteq \bar{R}^*(Y_2)$.

$L^*_9$. $R^*(Y) \subseteq R^*(R^*(Y))$.

$U^*_1$. $R^*(Y) = (\bar{R}^*(Y))^c$.

$U^*_2$. $\bar{R}^*(\emptyset) = \emptyset$.

$U^*_3$. $R^*(Y_1 \cup Y_2) = \bar{R}^*(Y_1) \cup \bar{R}^*(Y_2)$.

$U^*_4$. $\bar{R}^*(Y_1 \cap Y_2) \subseteq R^*(Y_1) \cap R^*(Y_2)$.

$U^*_5$. $Y_1 \subseteq Y_2 \Rightarrow \bar{R}^*(Y_1) \subseteq \bar{R}^*(Y_2)$.

$U^*_9$. $(\bar{R}^*(Y))^c \subseteq \bar{R}^*(Y)$.

Proof. By the duality of approximation operators, we only need to prove the properties $L^*_1 - L^*_5$ and $L^*_9$.

$L^*_1$. Since $\forall y \in V$, $V(y) = 1$ and $G(y) \subseteq V$, then $\min\{V(z) \mid z \in G(y)\} = 1$. Thus $R^*(V)(y) = \min\{V(z) \mid z \in G(y)\} = 1$. Therefore $R^*(V) = V$.

$L^*_2$. Since $\forall y \in V$, $V(y) = 1$ and $G(y) \subseteq V$, then $\max\{V(z) \mid z \in G(y)\} = 1$. Thus $\bar{R}^*(V)(y) = \max\{V(z) \mid z \in G(y)\} = 1$. Therefore $\bar{R}^*(V) = V$.

$L^*_3$. Since $\forall y \in V$, $Y_1 \cap Y_2(y) = \min\{Y_1(z) \cap Y_2(z) \mid z \in G(y)\}$, then $R^*(Y_1 \cap Y_2)(y) = \min\{Y_1(z) \cap Y_2(z) \mid z \in G(y)\}$, and $\min\{Y_1(z) \cap Y_2(z) \mid z \in G(y)\}$ and $\min\{Y_1(z) \cap Y_2(z) \mid z \in G(y)\}$ are referred to as the lower and upper approximations of $Y$.

$L^*_4$. Since $\forall y \in V$,

\[
R^*(Y_1 \cup Y_2)(y) = \min\{Y_1(z) \cup Y_2(z) \mid z \in G(y)\}
\]

$\bar{R}^*(Y_1 \cup Y_2)(y) = \max\{Y_1(z) \cup Y_2(z) \mid z \in G(y)\}$

From (1) and (2) we get

\[
R^*(Y_1 \cup Y_2)(y) = \max\{R^*(Y_1)(y), R^*(Y_2)(y)\}.
\]

Hence $R^*(Y_1 \cup Y_2) \supseteq R^*(Y_1) \cup R^*(Y_2)$.

$L^*_5$. Since $Y_1 \subseteq Y_2$, then $\forall y \in V$, $Y_1(y) \subseteq Y_2(y)$.

$L^*_9$. According to Lemma 3.1, if $z \in G(y)$, then $G(z) \subseteq G(y)$. Thus $R^*(R^*(Y))(y) = \min\{R^*(Y)(z) \mid z \in G(y)\}$.

From (1) and (2) we get

\[
R^*(Y_1 \cup Y_2)(y) \supseteq \max\{R^*(Y_1)(y), R^*(Y_2)(y)\}.
\]

Hence $R^*(Y) \subseteq R^*(R^*(Y))$.

Remark 5.1. If $R \in \mathcal{P}(U \times V)$ is a binary relation in a revised (two-universe) approximation space $(U, V, R)$, then the following properties do not hold for all $Y \in \mathcal{F}(V)$:

$L^*_6$. $R^*(\emptyset) = \emptyset$.

$L^*_7$. $R^*(Y) \subseteq Y$.

$L^*_8$. $Y \subseteq \bar{R}^*(R(Y))$.

$L^*_9$. $\bar{R}^*(Y) \subseteq R^*(R(Y))$.

$L^*_9$. $R^*(Y) \subseteq \bar{R}^*(R(Y))$.

$L^*_9$. $R^*(Y) \subseteq R^*(R(Y))$.

$L^*_9$. $R^*(Y) \subseteq \bar{R}^*(R(Y))$.

The following example shows Remark 5.1.

Example 5.1. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ and $R \in \mathcal{P}(U \times V)$ be a binary relation defined as:
<table>
<thead>
<tr>
<th>y_1</th>
<th>y_2</th>
<th>y_3</th>
<th>y_4</th>
<th>y_5</th>
<th>y_6</th>
<th>y_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>y_1</td>
<td>y_2</td>
<td>y_3</td>
<td>y_4</td>
<td>y_5</td>
<td>y_6</td>
</tr>
<tr>
<td>x_1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x_2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x_3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x_4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x_5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x_6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If Y is a fuzzy subset of V defined as:
Y(y_1) = 0.2, Y(y_2) = 0.5, Y(y_3) = 0.1, Y(y_4) = 0.7, Y(y_5) = 0.3, Y(y_6) = 0.8, Y(y_7) = 0.4, then we have

<table>
<thead>
<tr>
<th>y_1</th>
<th>y_2</th>
<th>y_3</th>
<th>y_4</th>
<th>y_5</th>
<th>y_6</th>
<th>y_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>R^*(Y)(y)</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>R^<em>(R^</em>(Y))(y)</td>
<td>0.2</td>
<td>0.3</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>R^*(Y)(y)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.8</td>
</tr>
<tr>
<td>R^<em>(R^</em>(Y))(y)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>R^*(∅)(y)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence we have R^*(∅) ≠ ∅, R^*(Y) ≠ Y, R^*(Y) ⊈ Y, Y ⊈ R^*(Y), Y ⊈ R^*(R^*(Y)), R^*(Y) ⊈ R^*(R^*(Y)), R^*(R^*(Y)) ⊈ Y, R^*(R^*(Y)) ⊈ R^*(R^*(Y)), R^*(Y) ⊈ R^*(R^*(Y)) and R^*(Y) ⊈ R^*(Y), i.e., L_{L^*}, L_{L^*}, L_{L^*}, L_{L^*}, L_{L^*}, L_{L^*}, L_{L^*}, U_{L^*} and U_{L^*} do not hold.

**Proposition 5.2** In a revised (two-universe) approximation space (U, V, R) with inverse serial relation R, the approximation operators have the following properties for all Y ∈ F(V):

L_{L^*} : R^*(∅) ≠ ∅.
L_{L^*} : R^*(Y) ⊈ Y.
U_{L^*} : R^*(Y) = Y.
U_{L^*} : Y ⊈ R^*(Y).
L_{L^*} : R^*(Y) ⊈ R^*(Y).
L_{L^*} : R^*(Y) ⊈ R^*(Y).

**Proof.** The proof comes from Definition 5.1 and Lemma 3.2.

**Remark 5.2.** If R ∈ P(U × V) is an inverse serial relation in a revised (two-universe) approximation space (U, V, R), then the following properties do not hold for all Y ∈ F(V):

L_{L^*} : Y ⊈ R^*(R^*(Y)).
U_{L^*} : R^*(R^*(Y)) ⊈ Y.
U_{L^*} : R^*(R^*(Y)) ⊈ R^*(Y).

The following example shows Remark 5.2.

**Example 5.2.** Let U = {x_1, x_2, x_3, x_4, x_5, x_6}, V = {y_1, y_2, y_3, y_4, y_5, y_6, y_7} and R ∈ P(U × V) be an inverse relation defined as:

<table>
<thead>
<tr>
<th>y_1</th>
<th>y_2</th>
<th>y_3</th>
<th>y_4</th>
<th>y_5</th>
<th>y_6</th>
<th>y_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>y_1</td>
<td>y_2</td>
<td>y_3</td>
<td>y_4</td>
<td>y_5</td>
<td>y_6</td>
</tr>
<tr>
<td>x_1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x_2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x_3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x_4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x_5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>x_6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If Y is a fuzzy subset of V defined as:
Y(y_1) = 0.2, Y(y_2) = 0.5, Y(y_3) = 0.1, Y(y_4) = 0.7, Y(y_5) = 0.3, Y(y_6) = 0.8, Y(y_7) = 0.4, then we have

<table>
<thead>
<tr>
<th>y_1</th>
<th>y_2</th>
<th>y_3</th>
<th>y_4</th>
<th>y_5</th>
<th>y_6</th>
<th>y_7</th>
</tr>
</thead>
<tbody>
<tr>
<td>R^*(Y)(y)</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>R^<em>(R^</em>(Y))(y)</td>
<td>0.2</td>
<td>0.3</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>R^*(Y)(y)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.8</td>
</tr>
<tr>
<td>R^<em>(R^</em>(Y))(y)</td>
<td>0.2</td>
<td>0.5</td>
<td>0.7</td>
<td>0.7</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>R^*(∅)(y)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Hence we have Y ⊈ R^*(R^*(Y)), R^*(Y) ⊈ Y, R^*(Y) ⊈ R^*(R^*(Y)) and R^*(R^*(Y)) ⊈ R^*(Y), i.e., L_{L^*}, U_{L^*}, L_{L^*} and U_{L^*} do not hold.

**Proposition 5.3** In a revised (two-universe) approximation space (U, V, R) with strong inverse serial relation R, the approximation operators have the following properties for all Y ∈ F(V):

L_{L^*} : Y ⊈ R^*(R^*(Y)).
L_{L^*} : R^*(Y) ⊈ R^*(R^*(Y)).
U_{L^*} : R^*(R^*(Y)) ⊈ Y.
U_{L^*} : R^*(R^*(Y)) ⊈ R^*(Y).

**Proof.** L_{L^*}. Since R is strong inverse serial, then z ∈ G(y) implies that G(z) = G(y) for all y, z ∈ V. Thus

\[ R^*(R^*(Y))(y) = \min \{ R^*(Y)(z) : z \in G(y) \} \]

= \min \{ \max \{ Y(w) : w \in G(z) \} : z \in G(y) \}

= \max \{ \min \{ Y(w) : w \in G(y) \} : z \in G(y) \}

= R^*(Y)(y)

Therefore R^*(Y) ⊈ R^*(R^*(Y)).

L_{L^*}. The proof is directly from U_{L^*} and L_{L^*}.

For U_{L^*} and U_{L^*} the proofs are similar.

**6 Conclusion**

In this paper we presented a new definition of the lower approximation and upper approximation on two
universes through the use of the intersection of right neighborhoods. All the properties of rough sets have been simulated by employing this notion. It has been revealed in this proposed model that the properties of Pawlak approximation space are attainable if the relation is strong inverse serial. Finally, this notion has been expanded for the rough fuzzy sets and the properties were the same reached by Dubois[4].

References

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